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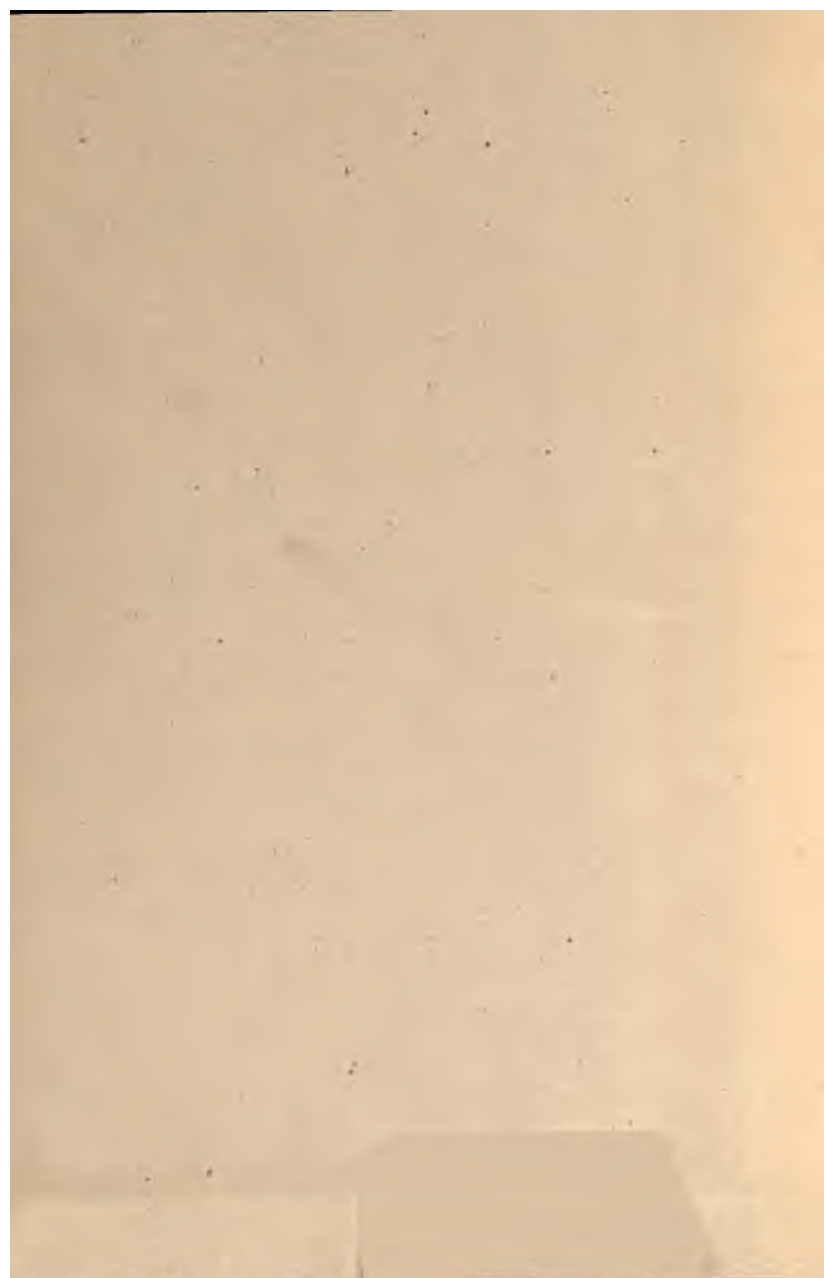


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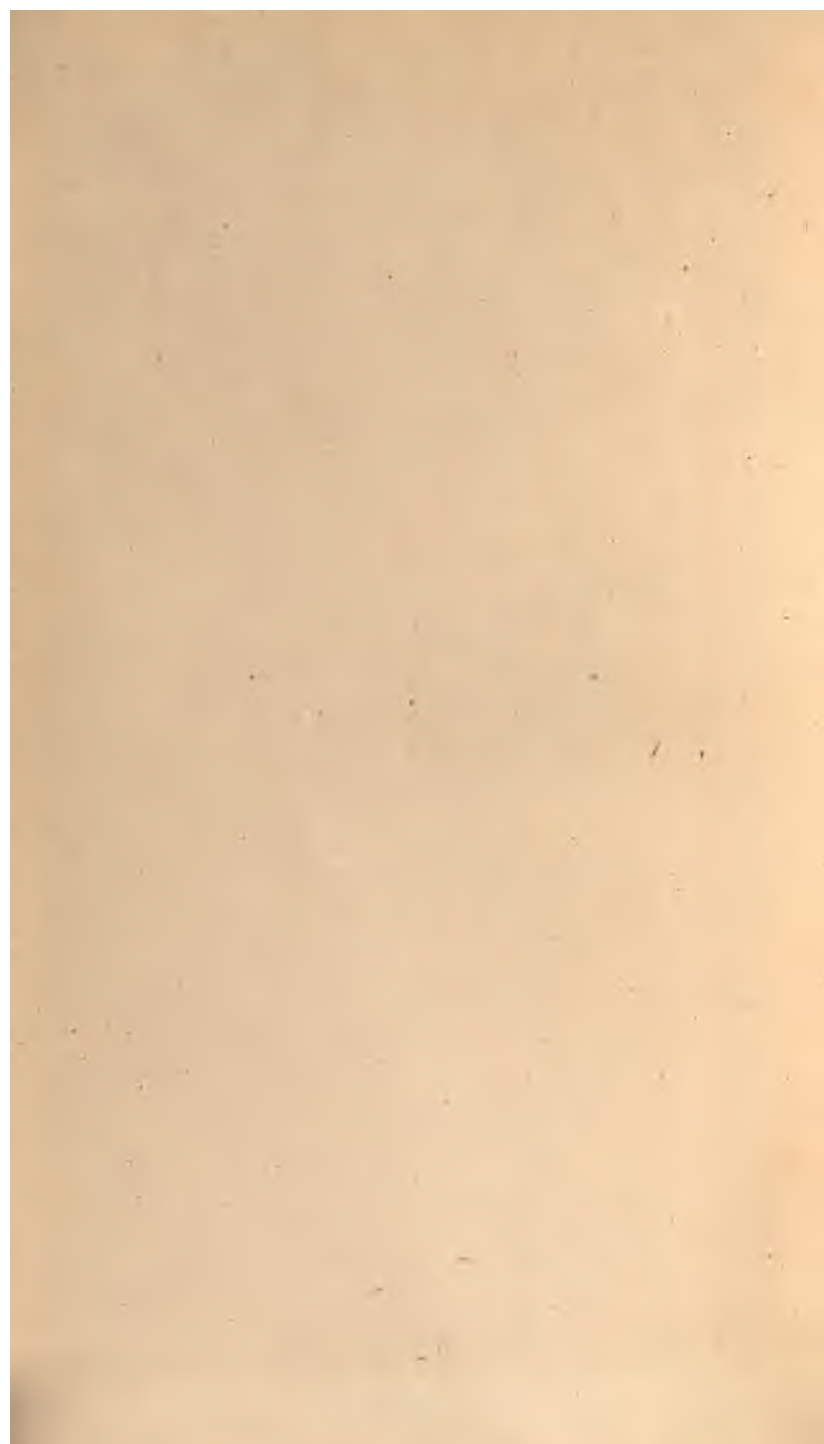
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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

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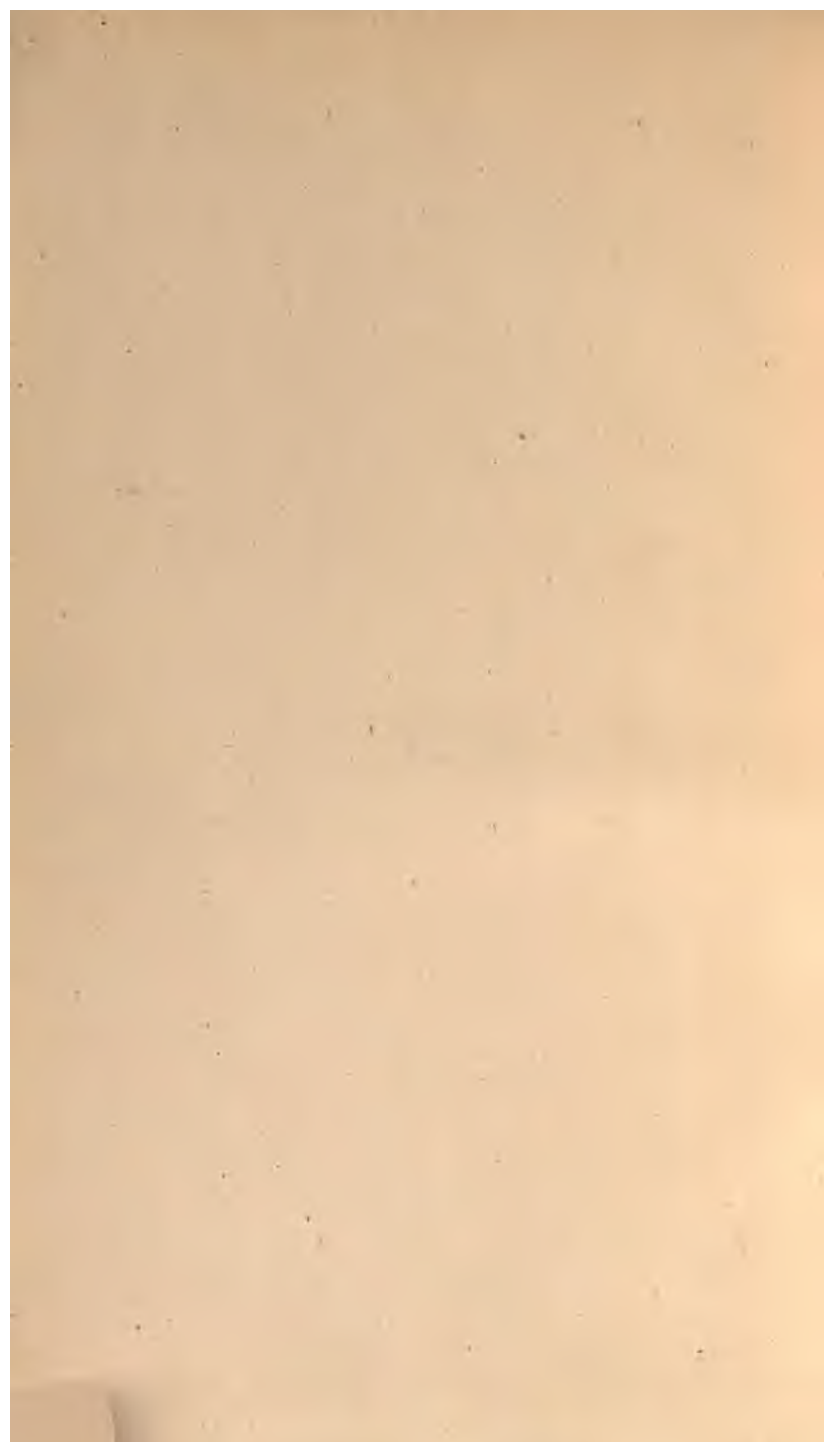
VOL. XXXVII.

LONDON

C. F. HODGSON & SON, GOUGH SQUARE,

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CONTENTS.

Mathematical Papers, &c.

No.	Page
179. Probability and Listerism. No. 1. By the Editor.	59
180. Probability and Listerism. No. 2. By the Editor.	77
181. Note on Egyptian Arithmetic. By T. Muir, F.R.S.E.	82
182. Note on Question 1621. By the Editor	87
183. Probability and Listerism. No. 3. By Elizabeth Blackwood. 92	
184. On the Atripthaloid and Atripthalid of Dr. Haughton. (Questions 6977, 7003, 7062.) By Professor Townsend, F.R.S.	102

Solved Questions.

2586. (The Editor.)—P is a given point outside or inside a circle whose centre is O; find the locus of another point R inside or outside the circle, such that, if P, R be joined, and lines be drawn through P, R to cut the circle in A, B and C, D, we may have

$$PR^2 = CR \cdot RD - AP \cdot PB. \dots\dots\dots 50$$

3246. (Professor Siverly, M.A.)—Two equal ellipses, whose axes are in the ratio of m to n , and transverse axes in the same straight line, and the distance between their centres d , intersect one another; find their axes when the three spaces formed by the intersecting ellipses are equal.... 121

4470. (The Editor.)—A heavy uniform beam (AB) moves freely about a hinge at A, and an elastic string is attached to the extremity B, and fixed at a point C, in the same horizontal line as A, at a distance (AC) equal to the length of the beam. The natural length of the elastic string is equal to half that of the beam, and its elasticity is such that the weight of the beam would stretch it to twice its natural length. Find the angle which the string makes with the horizon when the system is in equilibrium..... 39

4626. (Professor Malet, M.A.)—Given three conics passing through the same four points S, S' and S—kS', if through any point of S—kS' we draw two lines L and M, —L intersecting S in the points a , b , and M intersecting S' in the points A, B; then, if the lines Aa and Bb intersect S' again in the points C and D, prove that the locus of the intersection of the lines (C, D) and (a , b) is the conic S—kS'. 92

5052. (Professor Sylvester, F.R.S.)—Defining the *Bipotential* of an element of volume at a point-pair to mean the mass of the element multiplied by the product of the inverse distances of the element from the two points of the pair, and the bipotential of a volume to mean the sum of the bipotentials of its elements, prove that (1) the bipotential of a given uniform spherical shell in respect to an external point-pair is a function only of the bipotential at that point-pair of a unit element at the centre and the angle which the line joining the two points subtends at the centre; (2) the same proposition is true for the bipotential of the shell in respect to a point-pair lying entirely within the shell, but not for a point-pair of which one point lies within and the other outside the shell; (3) if a plane bipotential is defined as above, save that the product of the logarithms of the two distances is substituted in place of the product of their inverses, then the difference between the bipotential of a circular annulus at any external point-pair and of the annulus concentrated at its centre at the same point-pair depends only on the product of the distances of the two points from the centre and the angle which the line joining them subtends at the centre. 23

5099. (R. F. Scott, M.A.)—If $2a$, $2b$, $2c$ are the axes of an ellipsoid, d the distance from the centre to any point on the surface, and p the perpendicular from the centre on the tangent at that point, prove that the length of the normal chord at the point is

$$2a^2b^2c^2 + p[b^2c^2 + c^2a^2 + a^2b^2 - p^2(a^2 + b^2 + c^2) + d^2p^2]. \dots\dots 122$$

5244. (Professor Cayley, F.R.S.)—Writing for shortness

$$\begin{aligned} L &= \beta^2\gamma^2 - a^2\delta^2, & M &= \gamma^2a^2 - \beta^2\delta^2, & N &= a^2\beta^2 - \gamma^2\delta^2, \\ F &= a^2 + \delta^2 - \beta^2 - \gamma^2, & G &= \beta^2 + \delta^2 - \gamma^2 - a^2, & H &= \gamma^2 + \delta^2 - a^2 - \beta^2, \\ & & \Delta &= a^2 + \beta^2 + \gamma^2 + \delta^2; \end{aligned}$$

show that the equation

$$\begin{aligned} LMN(x^4 + y^4 + z^4 + w^4) + MN(F\Delta + 2L)(y^2z^2 + x^2w^2) \\ + NL(G\Delta + 2M)(x^2x^2 + y^2w^2) + LM(H\Delta + 2N)(x^2y^2 + z^2w^2) \\ - 2a\beta\gamma\delta FGH\Delta xyzw = 0 \end{aligned}$$

belongs to a 16-nodal quartic surface, having the nodes

$$\begin{array}{l} x = a \quad \left| \begin{array}{cccc} a & \beta & \beta & \beta \\ \beta & a & -a & -a \\ \gamma & -\gamma & \delta & -\delta \\ \delta & \delta & -\gamma & -\gamma \end{array} \right| \quad \left| \begin{array}{cccc} \beta & \beta & \gamma & \gamma \\ a & \delta & -\delta & -\delta \\ \alpha & -a & a & -a \\ \beta & \beta & -\beta & -\beta \end{array} \right| \quad \left| \begin{array}{cccc} \gamma & \gamma & \delta & \delta \\ -\delta & -\delta & \gamma & -\gamma \\ -\beta & -\beta & \alpha & -a \\ \alpha & -a & -\alpha & -a \end{array} \right| \\ y = \beta \quad \left| \begin{array}{cccc} a & \beta & \beta & \beta \\ \beta & a & -a & -a \\ \gamma & -\gamma & \delta & -\delta \\ \delta & \delta & -\gamma & -\gamma \end{array} \right| \quad \left| \begin{array}{cccc} \beta & \beta & \gamma & \gamma \\ a & \delta & -\delta & -\delta \\ \alpha & -a & a & -a \\ \beta & \beta & -\beta & -\beta \end{array} \right| \quad \left| \begin{array}{cccc} \gamma & \gamma & \delta & \delta \\ -\delta & -\delta & \gamma & -\gamma \\ -\beta & -\beta & \alpha & -a \\ \alpha & -a & -\alpha & -a \end{array} \right| \\ z = \gamma \quad \left| \begin{array}{cccc} a & \beta & \beta & \beta \\ \beta & a & -a & -a \\ \gamma & -\gamma & \delta & -\delta \\ \delta & \delta & -\gamma & -\gamma \end{array} \right| \quad \left| \begin{array}{cccc} \beta & \beta & \gamma & \gamma \\ a & \delta & -\delta & -\delta \\ \alpha & -a & a & -a \\ \beta & \beta & -\beta & -\beta \end{array} \right| \quad \left| \begin{array}{cccc} \gamma & \gamma & \delta & \delta \\ -\delta & -\delta & \gamma & -\gamma \\ -\beta & -\beta & \alpha & -a \\ \alpha & -a & -\alpha & -a \end{array} \right| \\ w = \delta \quad \left| \begin{array}{cccc} a & \beta & \beta & \beta \\ \beta & a & -a & -a \\ \gamma & -\gamma & \delta & -\delta \\ \delta & \delta & -\gamma & -\gamma \end{array} \right| \quad \left| \begin{array}{cccc} \beta & \beta & \gamma & \gamma \\ a & \delta & -\delta & -\delta \\ \alpha & -a & a & -a \\ \beta & \beta & -\beta & -\beta \end{array} \right| \quad \left| \begin{array}{cccc} \gamma & \gamma & \delta & \delta \\ -\delta & -\delta & \gamma & -\gamma \\ -\beta & -\beta & \alpha & -a \\ \alpha & -a & -\alpha & -a \end{array} \right| \end{array}$$

and the sixteen singular tangent planes represented by the equations $(\alpha, \beta, \gamma, \delta)(x, y, z, w) = 0$, &c. 74

5337. (J. J. Walker, M.A.)—Show how to transform two binary cubics u, v by the same linear substitutions into $(a, b, c, d)\tilde{Q}(x, y)^3$, and, to a factor $(a, -b, c, -d)\tilde{Q}(x, y)^3$ respectively, when this transformation is possible, viz., when the relation $\Delta\theta'^2 - \Delta'\theta^2 = 0$ is satisfied, the discriminant of $u + \lambda v$ being written $(\Delta, \theta, \phi, \theta', \Delta'\tilde{Q}(1, \lambda))^4$ 30

5397. (Elizabeth Blackwood).—If we take n positive random quantities $x_1, x_2, x_3, \dots, x_n$ between 0 and a , but with the *a priori* knowledge that all values of x_r between 0 and a are equally likely to be taken;

show that the chance that x_m is the greatest of the n quantities is $2m + n(n+1)$ 82

5527. (W. S. B. Woolhouse, F.R.A.S.)—Determine, by a simple construction, the size of the equilateral triangle from or into which a triangle can be orthogonally projected. 64

5556. (J. Hammond, M.A.)—Prove that the integral

$$I \equiv \int_0^1 \int_0^1 (u-v)^{2m} (uv) \quad du \, dv = \frac{1}{m+n} B(2m+1, n). \quad \dots\dots 57$$

5649. (S. Tebay, B.A.)—From a bag containing n cards, a card is drawn but not replaced; then a card is drawn and replaced i times in succession, the sum of the numbers drawn being s ; required the probability that the x th card was first drawn 52

5688. (Professor Sylvester, F.R.S.)—A tree proceeds from a fixed root, and is supposed subject to the law of bifurcation at each joint. Neglecting all distinctions founded on magnitude or direction, and putting u_n for the number of distinguishable forms of arborescence, when n is the number of joints, prove that, if

$$\phi x = -1 + x + u_1 x^2 + u_2 x^3 + u_3 x^4 + \dots, \text{ then } \phi(x^2) + (\phi x)^2 = -2x \dots 90$$

5694. (Professor Lloyd Tanner, M.A.)—Prove that every power of n is of the form $k(1+n+n^2+\dots+n^r)+n'$, where n' is 1, n , n^2 , ... or n^r . For instance, every power of 3 is of one of the forms $4k \pm 1$; or it is one of the forms $13k+1$, $13k+3$, $13k-4$; and so on. 25

5803. (J. J. Walker, M.A.)—Prove that (1) the condition that the vector $la+m\beta$ shall be terminated by the circumference through the origin and the ends of the vectors α, β , is

$$2lmSa\beta + l(1-l)T^2\alpha + m(1-m)T^2\beta = 0;$$

and (2) the condition for the circumference through the centres of the lines joining the ends of the vectors α, β , $la+m\beta$, containing the origin, is

$$(1-l^2-m^2)Sa\beta + lm(T^2\alpha + T^2\beta) = 0. \quad \dots\dots\dots 46$$

5919. (S. Tebay, B.A.)—Prove that the locus of the centre of a circle that cuts from a given triangle the greatest possible area is the rectangular hyperbola $(b^2-c^2)a^2 + (c^2-a^2)\beta^2 + (a^2-b^2)\gamma^2 = 0$ 30

5951. (D. Edwardes.)—If A, B, C are the angles of a triangle, and if $\alpha = \frac{\sin 2B + \sin 2C}{2 \cos 2A - 1}$, with similar expressions for β and γ , prove that $\alpha + \beta + \gamma = 4\alpha\beta\gamma$ 39

5997. (W. J. C. Sharp, M.A.)—Prove that the catalecticant of the binary $2n$ -ic is the discriminant of an $(n+1)$ -ary quadratic; and hence deduce the properties of the catalecticant. 33

6071. (Professor Matz, M.A.)—A point is taken at random in the diameter of a semicircle, a second point taken at random on the curve, and these points are joined with one end of the diameter; prove (1) that the mean area of the triangle thus formed is $\frac{r^2}{\pi}$; and (2) find the chance that one of these triangles is less than $\frac{1}{4}\pi r^2$ 121

6105. (G. Heppel, M.A.)—A and B are circulating decimals, each with a recurring period of eleven figures, and $B = (A + 2)^{-1}$; find A and B..... 27

6117. (The Editor.)—If e be the eccentricity of an ellipse whose major semi-axis is 1, and if E be the length of its quadrant, prove that

$$I \equiv \int_0^{\pi} \frac{e E de}{(1-e^2)(a^2-e^2)^{\frac{1}{2}}} = \frac{\pi a}{2(1-a^2)^{\frac{1}{2}}}, \dots\dots\dots 49, 111$$

6135. (W. S. B. Woolhouse, F.R.A.S.)—Find (1) the average area of triangles drawn on the surface of a given rectangle, and having one of the three sides parallel to a given line; and (2) the directions of the given line when the average area is a maximum and a minimum. 109

6138. (F. Morley, B.A.)—If A, B, C are fixed points, and forces are denoted by OA, OB, OC; find (1) the locus of O when these forces have a resultant fixed in (α) magnitude, (β) direction, (γ) magnitude and direction; and (2) extend the problem to tetrahedra..... 48

6160. (Professor Wolstenholme, M.A.)—Two fixed conjugate semi-diameters of an ellipse are CA, CB; P is any other point on the ellipse; A', B', P' are the corresponding points on the auxiliary circle; and A'a, B'b, P'p are chords of the auxiliary circle perpendicular to the major axis. Straight lines PL, PM are drawn parallel to CA', CB', and terminated by CA, CB; and straight lines Pl, Pm parallel to Ca, Cb, and also terminated by CA, CB respectively; prove that LM, lm are of constant lengths, and are perpendicular respectively to CP', Cp. Their lengths are $(a \pm b) \tan ACB$. 36

6176, 6708, & 7009. (The Editor.)—An iron hoop of radius t is thrown at random on (1) a round table of radius r , (2) a triangular table of inscribed radius r and semiperimeter s ; show that the probabilities (p_1, p_2) that the hoop will rest wholly on the table, and (q_1, q_2) that it will not fall off, are, respectively,

$$p_1 = \left(\frac{r-t}{r+t}\right)^2, \quad q_1 = \left(\frac{r}{r+t}\right)^2, \quad p_2 = \frac{(r-t)^2 s}{(r+2t)rs + \pi r t^2}, \quad q_2 = \frac{rs - \frac{1}{2}\pi t^2}{(r+2t)s + \pi t^2};$$

(3) if the hoop were a continuous disc (a florin for example) the probabilities would be the same, with the omission of $-\frac{1}{2}\pi t^2$ in the numerator of q_2 ; and (4) if $t = r$ and the triangle in (2) be equilateral, the probabilities are

$$p_1 = 0, \quad q_1 = \frac{1}{4}, \quad p_2 = 0, \quad q_2 = \frac{3\sqrt{3}}{9\sqrt{3} + \pi} = .277. \dots\dots\dots 92$$

6253. (R. Williamson, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle..... 63

6266. (Rev. C. Taylor, D.D.)—Determine the envelop of the circle through the three points in which a triad of concurrent lines drawn from the vertices of a given triangle meet the opposite sides. 21

6262. (Rev. H. G. Day, M.A.)—If a line of length c is divided into n parts, $x, y, z, \dots$, by $(n-1)$ points taken at random in it, prove that the average values (1) of $xyz \dots$, (2) of $x_n^a y_n^b z_n^c \dots$, are

$$\frac{(n-1)!}{(2n-1)!} c^n, \quad \frac{(n-1)! \alpha! \beta! \gamma! \dots}{(n-1+\alpha+\beta+\gamma \dots)!} c^{\alpha+\beta+\gamma \dots}, \dots\dots\dots 101$$

6328. (G. J. Griffiths, M.A.)—A rough uniform circular hoop, of radius a and mass M , rests in a vertical plane on a perfectly rough horizontal plane, and an insect of mass m suddenly begin to move from the extremity of a horizontal diameter, and crawls along the hoop with uniform relative velocity V ; prove that, when it arrives at the highest point of the hoop, the angular velocity ω of the hoop is given by the equation

$$(M + 2m)(V - a\omega)^2 + mga = \left(\frac{2M}{2M + m}\right)^2 (M + m)V^2. \quad \dots\dots 31$$

6344. (Professor Wolstenholme, M.A.)—Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4 (1 + 2 \cos^2 x)} F(\sin^2 x) dx = \frac{1}{2} \int_0^{\pi} F(\sin^2 x) dx \quad \dots\dots\dots (1),$$

$$\int_{-\infty}^{\infty} \frac{\sin^6 x}{15 - 15 \sin^2 x + 2 \sin^4 x} \frac{F(\sin^2 x)}{x^6} dx = \frac{1}{12} \int_0^{\pi} F(\sin^2 x) dx \quad \dots\dots\dots (2).$$

..... 100

6360. (J. W. Russell, M.A.)—A string AC is stretched tight, and a point B in it passes through a smooth ring; particles of equal mass attached to A and C are projected in the same plane and perpendicular to BA and to CA with equal angular momenta about B, and are acted on by no forces except the tension of the string; show that the two particles will describe the same path..... 34

6364. (W. E. Wright, B.A.)—If A, B, C be the angles of a plane triangle, find the minimum value of $\cot A + \cot B + \cot C$ 52

6369. (J. L. McKenzie, B.A.)—If $s_r = a_1^r + a_2^r + \dots + a_n^r$, where r is any positive integer, prove (1) that

$\begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ s_{m-1} & s_{m-2} & \dots & m-1 & \\ s_m & s_{m-1} & \dots & s_1 & \end{vmatrix}$	$= (-1)^n a_1 a_2 a_3 \dots a_n n!$, when $m = n$; and (2) that the determinant vanishes for any value of m greater than n 39
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6397. (H. L. Orchard, M.A.)—A cube and a tetrahedron are inscribed in two equal spheres; show that the ratio of the sums of their edges can be expressed as a simple trigonometrical function..... 51

6434. (R. Knowles, B.A., L.C.P.)—If through a point P in a conic a tangent be drawn at right angles to a chord OQ, prove that (1) the chords OP, QP will make angles with the normal at P, such that the product of their tangents is constant; and (2) if the conic be an equilateral hyperbola, OP, QP will be at right angles. 35

6455. (By G. F. Walker, M.A.)—A number of uniform circular discs of radius a , but of any masses, are freely movable, in a vertical plane, about their centres, which are fixed in a horizontal line at distances $4a$ apart. A fine rough string, of indefinite length, having two equal particles of mass m at its extremities, is laid over these circles, and uniform circular discs, each of radius a and mass $2m$, are laid on the string so as to hang between the other circles, the parts of the string not in contact with a circle being vertical. Prove that, if the system be in motion under the action of gravity, all its parts will move uniformly so long as the centres of all the discs $2m$ are below the line of fixed centres. 47

6537. (Professor Wolstenholme, M.A.)—At any point of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, r denotes the central distance, p the perpendicular from the centre on the tangent plane, n the length of the chord intercepted by the ellipsoid on the normal, and ϕ the angle between the tangent planes at the ends of this chord; prove that

$$np = \frac{2a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2 - p^2(a^2 + b^2 + c^2 - r^2)} \dots\dots\dots (1),$$

$$2a^2b^2c^2 \tan^2 \phi = n^2p^3(2p - n) - (2a^2 - np)(2b^2 - np)(2c^2 - np) \dots\dots\dots (2).$$

53

6578. (G. F. Walker, M.A.)—A fine circular tube, of uniform bore and of radius a , in a vertical plane, contains columns of n fluids or densities $\rho_1, \rho_2, \dots, \rho_n$ respectively, the respective columns subtending angles $2a_1, 2a_2, \dots, 2a_n$ at the centre of the circle. Show that, if θ be the angle, the radius to the free surface of the fluid of density ρ_1 makes with the vertical when the system is in equilibrium under the action of gravity,

$$\tan \theta = \sum_{r=1}^{r=n} \rho_r \sin a_r \sin (2s_r - a_r) + \sum_{r=1}^{r=n} \rho_r \sin a_r \cos 2s_r - a_r$$

where s_r denotes $a_1 + a_2 + \dots + a_r$. 36

6584. (R. Knowles, B.A., L.C.P.)—Show that the locus of the centres of circles that pass through a given point, and touch a given straight line, is a parabola. 123

6594. (The Editor.)—A troop of Horse Artillery, on the march from Woolwich to Chatham, met a company of Marines marching at the rate of 3 miles an hour towards Woolwich; 4 miles further on, and 4 hours after leaving Woolwich, they met some stragglers from the company hastening on at the rate of 4 miles an hour, which pace would just enable them to join the company as it marched into Woolwich: find (1) at what rate the troop was marching, (2) at what distance from Woolwich it met the company of Marines, and (3) interpret the negative result. 29

6602. (Professor Nash, M.A.)—Prove that (1) the radical axis of the circles of curvature at the extremities of any ordinate to a given diameter of a parabola passes through a fixed point, and (2) the locus of this point for different diameters is a parabola. 37

6652. (J. Hammond, M.A.)—Show that

$$\frac{1}{m} + \int_0^1 \int_0^1 e^{xy} (1-y)^{m-1} x^m dx dy = \int_0^1 e^x (1-x)^{m-1} dx \dots\dots\dots 29$$

6659. (Professor Townsend, F.R.S.)—In the irrotational strain of an incompressible lamina in the plane of its mass, if the potential and displacement line systems of the strain be two orthogonal systems of coaxial circles; show that, throughout the lamina, the actual molecular displacement and principal linear dilatation vary respectively, the former inversely as the product of the distances from the foci, and the latter directly as the distance from the centre and inversely as the product of the squares of the distances from the foci of the systems. 32

6674. (J. J. Walker, M.A.)—If τ', τ'', τ''' are vectors parallel to the tangents to the circumference passing through the terms of the vectors α, β, γ , at those points respectively; show that—

$$(1) \quad \tau' = b^2(\alpha - \beta) + c^2(\gamma - \alpha), \quad \tau'' = c^2(\beta - \gamma) + a^2(\alpha - \beta), \quad \tau''' = \dots,$$

- where $a = \tau(\beta - \gamma)$, $b = \tau(\gamma - \alpha)$, $c = \tau(\alpha - \beta)$;
and (2) that the vector parallel to the tangent at the term of $la + m\beta + n\gamma$
is $\tau = lr' + m\tau'' = n\tau'''$ 67
6708. (The Editor.) See 6176..... 92
6718. (R. Tucker, M.A.)—ACB is a right-angled triangle, C the
right angle. Rhombuses are inscribed having one side coincident with
the hypotenuse; find (a) the locus of their centroids, and show (b) that
the normal to the locus at C meets the curve again in the centroid of the
triangle. 118
6728. (S. Constable, M.A.)—If O, O_1, O_2, O_3 are the centres of the
inscribed and escribed circles of a triangle ABC; (α, β, γ) , $(\alpha', \beta', \gamma')$ the
points of contact of circles $(O), (O_1)$, &c.; $(\Delta, \Delta', \Delta'', \Delta''')$, $(\delta, \delta', \delta'', \delta''')$
the areas of the triangles $O_1O_2O_3, OO_2O_3$, &c., $\alpha\beta\gamma, \alpha'\beta'\gamma'$, &c.; prove
$$\delta = \frac{S^2}{\Delta}, \delta' = \frac{S^2}{\Delta'}, \text{ \&c.}; \frac{1}{\delta'} + \frac{1}{\delta''} + \frac{1}{\delta'''} = \frac{1}{\delta}.$$
 35
6738. (Professor Crofton, F.R.S.)—Prove that
 $D^n f(xD) X = f(xD + n) D^n X$ 63
6743. (The Editor).—Find the probability that the perimeter of a
triangle whose corners are the centre of a given circle and two random
points within the circle, is less than three times the radius 24
6750. (The Rev. W. A. Whitworth, M.A.)—A pavement is made of
black and white tiles, the black being squares, and the white equilateral
triangles. Construct the pattern so that the areas of black and white may
be in the ratio of $\sqrt{3} : 4$ 38
6753. (R. Tucker, M.A.)—PSQ, PHR are two focal chords of an
ellipse inclined at an angle ω ; if the centre of the ellipse be the centroid
of the triangle PQR, prove that the area of the triangle is $\frac{1}{2}b^2 \tan \frac{1}{2}\omega$. 70
6757. (Professor Hudson, M.A.)—Prove that $\frac{1}{1 + \frac{1}{2} + \frac{1}{3} + \dots}$ has no more
than 18 places in its period, when converted into a circulating decimal. 44
6759. (H. G. Dawson.)—Show that (1) the equation of an ellipse is
 $\rho^{-\frac{1}{2}} = (ab)^{-\frac{1}{2}} \{a^2 \sin^2 \psi + b^2 \cos^2 \psi\}$, where ρ is the radius of curvature at
any point, and ψ the angle which tangent makes with axis of x ; and
hence (2) that the area enclosed between quadrant of ellipse, axes, and
evolute is $\frac{3(a^4 + b^4) + 2a^2b^2}{32ab} \pi$; and (3) if the equation of a curve be
 $\rho = a \cos^m \psi$, the area between the curve, the evolute, and the normals at
the points where $\psi = \frac{1}{2}\pi, \psi = 0$, is $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \frac{\pi a^2}{4}$ 51
6766. (Professor Cayley, F.R.S.)—Find the stationary and double
tangents of the curve $x^4 + y^4 + z^4 = 0$ 64
6781. (R. Tucker, M.A.)—B, C, D are three points on an ellipse,
of centre O, the osculating circles at which countersect in a point A on the
curve; tangents at B, C, D intersect in B', C', D'; prove that these points
lie on a concentric ellipse (axes $2a, 2b$), and are situated with regard to
A' (the point where AO meets the new ellipse) in like manner as A, B,
C, D are. 122
6787. (Rev. W. A. Whitworth, M.A.)—If $3n$ cyphers be placed be-
tween the digits 3 and 7, prove (1) that the number thus formed is
divisible by 37; and if $3n + 1$ cyphers be placed between the digits 7 and
3, prove (2) that the number thus formed is divisible by 37. 55

6789. (John Griffiths, M.A.)—If x_1, x_2, x_3, x_4 are the roots of the equation
$$x^4 - \mu x^3 - \left(\frac{t^2}{e^2} + 1 - \frac{\mu^2}{4} \right) x^2 + \mu x + \frac{t^3}{e^4} - \frac{\mu^3}{4} = 0,$$

where t is a variable, and μ, e constants; show (1) that

$$\int \left(\frac{1 - e^2 x^2}{1 - x_1^2} \right)^{\frac{1}{2}} dx_1 + \int \left(\frac{1 - e^2 x^2}{1 - x_2^2} \right)^{\frac{1}{2}} dx_2 + \dots dx_3 + \dots dx_4 = 2t + e;$$

and (2), taking $\mu < 2$ and $e < 1$, deduce FAGNANI'S theorem with respect to two elliptic arcs. 26

6792. (C. Morgan, B.A.)—DAE is a triangle, having a right angle at A. From A lines AB, AC are drawn, making the angles DAB, DAC each equal to the angle E, and meeting DE in B and C respectively, show that $BA^2 : BE^2 = DC : EC$ 124

6796. (Professor Crofton, F.R.S.)—Prove that

$$e^{DxD} \cdot e^{hx} = \frac{1}{1-h} e^{\frac{hx}{1-h}}. \dots\dots\dots 26$$

6813. (W. J. C. Sharp, M.A.)—If a conic pass through the three points in which a triad of concurrent lines from the vertices of a triangle meet the opposite sides, the lines joining its other points of intersection with the sides to the opposite angles will be concurrent. If

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

be the equation to such a conic referred to the triangle as triangle of reference, $ax^2 + by^2 + cz^2 - 2fyz - 2gzx - 2hxy = 0$ breaks up into straight lines 21

6819. (R. Knowles, B.A., L.C.P.)—Show that the sum to n terms of the series $1 + 5 + 13 + 29 + 61 + \dots = 4(2^n - 1) - 3n$ 50

6821. (Professor Cavallin, M.A.)—A point P is taken at random in the area of a vertical circle, and another point Q at random in the segment below the horizontal chord through P; show that the probability that the time of descent for a particle down the straight line PQ is less than that from P down the straight line of quickest descent to the circle, is $\frac{1}{3}$ 27

6828. (Professor Townsend, F.R.S.)—A variable chord of a plane nodal cubic being supposed divided harmonically in every position, (a) by the two nodal tangents, (b) by any fixed pair of lines dividing harmonically the nodal angle; show that—

(a') It envelopes, in the former case, a conic; touching the two nodal tangents at their two points of intersection with the inflexional axis of the curve;

(b') It turns, in the latter case, round a point on the curve; the intersection, viz., of the tangents at the two points of meeting of the fixed pair of lines with the curve. 54

6839. (Professor Hudson, M.A.)—If α, δ be one pair of coordinates l, λ another, α, l being measured from the same origin, and ω the inclination of the planes of reference, prove that

$$\cos \omega = \frac{\sin \lambda \tan l + \sin \delta \tan \alpha}{\sin \lambda \tan \alpha + \sin \delta \tan l}. \dots\dots\dots 82$$

6843. (W. J. C. Sharp, M.A.)—If a particle be free to move in a fixed circular tube, centre O, and if the particle be subject to the action of two forces, one repelling and the other attracting, each varying as (distance)⁻ⁿ, and directed to two points F, F' which are reciprocal with respect to the circle; the particle will rest at any point of the tube if the forces at the unit of distance are as (OF)^{1/(n-1)} : (OF')^{1/(n-1)}..... 118

6844. (J. L. McKenzie, B.A.)—Show that the equation of the envelop of a circle passing through the cusp of a cardioid, and cutting the curve at right angles, is

$$27x^2y^2(x^2+y^2) + (x^2+y^2-2ax)^3 = 0. \dots\dots\dots 53$$

6848. (R. Knowles, B.A., L.C.P.)—AB, BC are two lines perpendicular to each other, A is a fixed point, and B moves along a given straight line; prove that the envelop of the locus of C is a parabola.... 56

6849. (Asutosh Mukhopadhyay.)—In a given triangle, the sum of the squares on the sides is equal to eight times the square on the line joining the vertex with the middle point of the base; prove that the triangle is right-angled. 50

6863. (W. J. C. Sharp, M.A.) — If the Cartesian coordinates of a point be each the same linear function of the corresponding coordinates of any number of points; prove that (1) the relation of the first point to the others is independent of the axes; and (2), if the sum of the coefficients be unity, also of the origin. 117

6866. (Rev. A. J. C. Allen, B.A.)—K is the orthocentre of a triangle ABC, and a second triangle A'B'C' is formed by drawing through AB, AC lines parallel to the opposite sides; two conics are drawn having K as a common focus, one inscribed in the triangle ABC, the other in A'B'C'; prove that the latus rectum of one is double that of the other..... 28

6867. (T. Muir, M.A., F.R.S.E.) — M. Landry has shown that $2^{64} + 1$ is divisible by $1071 \cdot 2^8 + 1$. Prove that, as a consequence of this, the numbers $1071^2 + 16777216^2$, $1071^4 + 256^4$, $1071^8 + 1^8$ are also so divisible 24

6875. (R. Tucker, M.A.)—The sides of a triangle are in arithmetical progression, and the cosine of the mean angle = $\frac{1}{10}$: (1) find the sides; and (2) prove that the difference between the greatest and least angles = $\cos^{-1} \frac{3}{4}$ 28

6889. (Professor Crofton, F.R.S.)—Prove that $D^r x^r F(x) = 1.2 \dots r \left[F(x) + xF'(x) + \frac{1}{2!} D^2 x^2 F'(x) + \frac{1}{3!} D^3 x^3 F''(x) + \dots (r+1) \text{ terms} \right]$.
..... 45

6891. (Professor Genese, M.A.)—If A, B, C be the foci of a Cartesian oval, and O the triple focus, prove that the length N of the normal from a point G in the axis is given by $N^2 = \frac{AG \cdot BG \cdot CG}{OG}$ 72

6892. (Professor Minchin, M.A.)—1. In the pure rolling of a circle on any plane curve, prove that the lengths of the roulettes described by any two inverse points bear a very simple ratio to each other. [This is a particular case of a general theorem.]

2. In the pure slipping (*i.e.*, without rotation) of any curve A on any other B, prove that the radius of curvature at any point of the locus traced out by any carried point is equal to the sum of the radii of curvature of A and B at their corresponding point of contact. 44

6893. (Professor Wolstenholme, M.A.)—A chord PQ of a given conic is drawn through a fixed point O; O' is another fixed point; and PP', QQ' are chords of the conic drawn through O'; prove that (1) P'Q' passes through a third fixed point O'' such that O, O', O'' are collinear; and (2) OO'' is divided harmonically by O' and the polar of O' with respect to the given conic. 22

6896. (Professor Cavallin, M.A.)—If p be the perpendicular from the curve of a sphere on the plane determined by three random points within the sphere; prove that the mean value of p^n is

$$\frac{1 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{(n+1)(n+6)(n+7)(n+8)(n+9)(n+10)} (2r)^n. \dots\dots\dots 85$$

6898. (The Editor.)—Show that the average area of all the circles that can be drawn through two random points so as to lie wholly within a given circle, is two-sevenths of the area of the circle. 43

6900. (Elizabeth Blackwood.)—If A, B, C, D are four candidates whose respective chances of passing a certain examination are a, b, c, d ; show that the chance that at least two out of the four will pass is $3abcd - 2\S abc + \Sigma ab$, where Σabc denotes combinations of a, b, c, d , three at a time, which, when $a = b = c = d$, becomes $6a^3 - 8a^3 + 3a^4$ 68

6908. (Sarah Marks.)—P is a point in any conic, PK the perpendicular therefrom on the directrix; with the usual notation, PS is produced to F such that SF = AX; the circle through PAF cuts the axis in M: prove that KPMS is a parallelogram. 26

6910. (J. Young, B.A.)—If two parallel chords of two concentric circles are at a given distance apart, prove that the difference of their lengths is a minimum when they subtend the same angles at the centre. 73

6915. (D. Edwardes.)—Upon the surface of a circle and so as to lie wholly within it, two circles, each of radius equal to one-third that of the given circle, are drawn at random; prove that the chance that they will not intersect is $3\sqrt{3} + 4\pi$ 47

6918. (A. McIntosh, B.A.)—Two tangents are drawn at right angles to an ellipse, p is the perpendicular from the centre of the ellipse on the chord of contact, and p' is the perpendicular from the intersection of the tangents on the same line; prove that $\frac{1}{pp'} = \frac{1}{a^2} + \frac{1}{b^2}$. What corresponds to this in the case of a quadric surface? 86

6919. (Professor Sylvester, F.R.S.)—If $u_{x+1} - u_x^2 + u_x - 1 = 0$, prove that $\Sigma \frac{1}{u_x} = \frac{1}{u_0 - 1} - \frac{1}{u_{x+1} - 1}$. Thus, suppose $u_0 = 5$; then $u_1 = 21$, $u_2 = 421$; and, if $x = 1$, we ought to find $\frac{1}{2} + \frac{1}{21} = \frac{1}{2} - \frac{1}{421}$, which is right. 42

6920. (Professor Townsend, F.R.S.)—Two forces in a common space being supposed to act along two non-intersecting right lines in the space; required the pitch of the equivalent wrench, in terms of the magnitudes of the forces, and of the distance and angle between their lines of direction in the space. 94

6923. (Professor Genese, M.A.)—A rigid right angle AMQ slides in a vertical plane with AM in a horizontal groove CS. A string fastened at S passes over a pulley at C, and is attached to a heavy particle Q free to slide in a groove MQ. Prove that, (1) if S be a fixed point and C attached to and moving with TM, Q will describe a rectangular hyperbola; (2) if C be fixed, and S is attached to AM, Q will trace a parabola..... 72

6926. (Professor Hadamard.)—Étant donnés un angle XSY et un point A dans son plan, on mène du point A une sécante quelconque AMN, et l'on divise la droite MN dans un rapport donné $m : n$ par le point P. On demande le lieu du point P. 81

6927. (Professor Minchin, M.A.)—The surface of still water is agitated by wave disturbances emanating from three given centres; assuming the amplitude of each of the disturbances to be constant, find the locus of points of zero disturbance (if such exist). 116

6928. (The Editor.)—Show that the values of x in the equations

$$\frac{1+xy}{a} = \frac{x+y^2}{b} = \frac{x^2+y}{c} = \frac{1-x^3}{a-cx} \dots\dots\dots (1),$$

are given by $(a^2-bc)x^4 + (c^2+ab)x^3 - 4acx^2 + (a^2+bc)x + (c^2-ab) = 0 \dots (2)$; and that, when $a=16$, $b=17$, $c=18$, equation (2) and its roots are

$$25x^4 - 298x^3 + 576x^2 - 281x - 26 = 0. \dots\dots\dots (3),$$

—079154299618, 1·060498774748, 1·283137095508, 9·655518429362. 62

6929. (Dr. Donald McAlister.)—Of 10 cases treated by Lister's method, 7 did well and 3 suffered from blood-poisoning; of 14 cases treated with ordinary dressings, 9 did well and 5 had blood-poisoning; what are the odds that the success of Lister's method was due to chance? ... 40

6930. (J. Griffiths, M.A.)—Show (1) that, if

$$\frac{a}{a'} \sin \theta \sin \theta' + \frac{b}{b'} \cos \theta \cos \theta' = 1, \text{ where } a^2 - b^2 = a'^2 - b'^2,$$

then $\int (a'^2 \cos^2 \theta + b'^2 \sin^2 \theta)^{\frac{1}{2}} d\theta - \int (a^2 \cos^2 \theta' + b^2 \sin^2 \theta')^{\frac{1}{2}} d\theta'$

$$= [(a \sin \theta - a' \sin \theta')^2 + (b \cos \theta - b' \cos \theta')^2]^{\frac{1}{2}} + \text{constant};$$

also (2) give a geometrical interpretation of the result; and (3) deduce therefrom GRAVES' theorem (SALMON's *Conics*, 6th ed., p. 377). 68

6934. (G. F. Walker, M.A.)—Solve the first set, and eliminate x, y, z from the second set, of the following equations:—

$$\begin{array}{l|l} (z+x-y)(x+y-z) = ax, & (z+x-y)(x+y-z) = ayz, \\ (x+y-z)(y+z-x) = by, & (x+y-z)(y+z-x) = bzx, \\ (y+z-x)(z+x-y) = cz, & (y+z-x)(z+x-y) = cxy. \end{array} \dots\dots\dots 114$$

6935. (R. Tucker, M.A.)—If $a+b+c=0$, prove that

$$\left(\frac{b^3-c^3}{a^3} + \frac{c^3-a^3}{b^3} + \frac{a^3-b^3}{c^3} \right) \left(\frac{a^3}{b^3-c^3} + \frac{b^3}{c^3-a^3} + \frac{c^3}{a^3-b^3} \right) = 36 - 4(a^3+b^3+c^3)(a^{-3}+b^{-3}+c^{-3}). \dots\dots\dots 46$$

6937. (J. Hammond, M.A.)—Find in how many different ways a different things can be given to b different people..... 67

6950. (C. Leudesdorf, M.A.)—Prove that, if

$$a [b^2x^2 + c^2y^2 + yz (b^2 + c^2 - a^2)]^{\frac{1}{2}} + b [c^2x^2 + a^2y^2 + xz (c^2 + a^2 - b^2)]^{\frac{1}{2}} \\ + c [a^2y^2 + b^2x^2 + xy (a^2 + b^2 - c^2)]^{\frac{1}{2}} = 0,$$

and $x + y + z = 1$, then either $a^2yz + b^2zx + c^2xy = 0$ or $a \pm b \pm c = 0 \dots 74$

6958. (Professor Hadamard, M.A.) — Etant données deux circonférences O et O' qui se coupent aux points A et B, on fait tourner autour du point A un angle de grandeur constante, dont les côtés coupent les circonférences en deux points M et N. On divise la droite MN dans un rapport donné $m : n$ par le point P, et on demande le lieu de ce point. 95

6961. (T. Muir, M.A., F.R.S.E.)—The Jacobian of the functions

$$\begin{array}{l|l} V_1 \equiv -ay - bz - eu - dv & V_4 \equiv cx + fy + hz - lv \\ V_2 \equiv ax & -ez - fu - gv & V_5 \equiv dx + gy + kz + lu, \\ V_3 \equiv bx + ey & -hu - kv \end{array}$$

being a zero-axial skew determinant of odd order, vanishes; find the relation connecting the functions..... 58

6963. (The Rev. H. G. Day, M.A.)—If three points be taken at random within a circle of radius c , prove that the mean area of the triangle whose corners are the random points is $\frac{35c^2}{48\pi}$ 120

6969. (R. Tucker, M.A.)—AP, AQ are two vertical chords of a parabola, which are complementally inclined to the axis, and ϕ , ϕ' are the angles made by the tangents at P, Q with AP, AQ; prove that

$$\cot \phi \sim \cot \phi' = 2\Delta APQ \div (\text{Latus Rectum})^2. \dots\dots\dots 114$$

6970. (Dr. Curtis.)—An ellipse is placed with its minor axis horizontal; find (1) the lines of quickest, or slowest, descent from any point on the major axis to the curve, or *vice versa*; (2) the condition that the quickest, or slowest, path should be along the major axis. 62

6972. (H. Stewart, M.C.P.)—Solve the equations—

$$x^2 + yz = a, \quad y^2 + zx = b, \quad z^2 + xy = c. \dots\dots\dots 62$$

6978. (Professor Sylvester, F.R.S.) — If i , j are any two positive integers and j not greater than i , prove that

$$(x-1)(x-2)\dots(x-i+1) - \frac{j}{i-j+1} x(x-1)\dots(x-i+2) + \frac{j(j-1)}{(i-j+1)(i-j+2)} \\ \times (x+1)x\dots(x-i+3) \\ \dots\dots + (-)^j \frac{j(j-1)\dots 1}{(i-j+1)(i-j+2)\dots i} (x+j-1)(x+j-2)\dots(x+j-i+1) \\ = \frac{i-j}{i} \frac{(x-1)(x-2)\dots(x-i)}{(x+j-i)}.$$

$$\text{Ex.}-(x-1)(x-2)-\frac{1}{2}x(x-1)=\frac{2}{3}\frac{(x-1)(x-2)(x-3)}{x-2}. \quad [i=3, j=1.]$$

$$(x-1)(x-2)(x-3)-\frac{3}{2}x(x-1)(x-2)+\frac{2}{3}\cdot\frac{1}{4}(x+1)x.(x-1) \\ =\frac{2}{3}\frac{(x-1)(x-2)(x-3)(x-4)}{x-2}. \quad [i=4, j=2.] \dots\dots 98$$

6980. (Professor Malet, M.A.)—If through a point P, situated within the surface of a given ellipsoid, a plane be drawn so as to cut the volume of the ellipsoid most unequally, prove that the plane is conjugate to the line joining P to the centre of the ellipsoid. 124

6982. (Professor Minchin, M.A.) — Determine the velocity at every point of a moving liquid, as a function of the coordinates, in such a manner that at every point in the liquid there shall be screw motion... 89

6984. (The Editor.)—Prove geometrically that the centre of one of the circles of contact (inscribed or escribed circles) of a triangle, the mid-point of one of the sides, and the mid-point of the line joining the opposite vertex with the point of contact of the circle with that side, are three points in the same straight line. 83

6986. (W. S. M'Cay, M.A.) — A lamina moves in its own plane, starting from any position and returning to the same position after one revolution; prove that (1) all straight lines that have enveloped curves of equal length (l) are tangents to the same circle in the lamina; (2) the length l is equal to the circumference of this circle; also (3) as the length l varies, the circles are concentric and have the same centre as KEMPE'S circles. 87

6987. (B. Reynolds, M.A.)—Show that (1) 8 queens can be placed on an ordinary chess-board, so that no two can take each other, in 12 essentially different classes, forming, with permutations, 92 different sets, of ways; (2) 7 queens can be placed on a 7-square board in 6 different classes, and 40 different sets, of ways, so as not to be able to take each other. 71

6988. (J. Griffiths, M.A.)—Show that,

$$x + \mu_1 y + \nu_1 z = a\mu_1^2 + b\nu_1^2, \quad x + \mu_2 y + \nu_2 z = a\mu_2^2 + b\nu_2^2, \quad x + \mu_3 y + \nu_3 z = a\mu_3^2 + b\nu_3^2, \\ \mu_2\mu_3 + \nu_2\nu_3 = -1 = \mu_2\mu_1 + \nu_2\nu_1 = \mu_1\mu_3 + \nu_1\nu_3, \text{ then } x = a + b. \dots 81$$

6992. (G. F. Walker, M.A.) — Three masses m_1 , m_2 , m_3 are fastened to a string which passes through a ring, and m_1 describes a horizontal circle as a conical pendulum, while m_2 and m_3 hang vertically; if m_3 drop off, prove that the instantaneous change of tension of the string is $\frac{gm_1m_3}{m_1+m_2}$ 95

6995. (E. W. Symons, M.A.)—Given a , b , c all positive, and

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 1,$$

prove that the least value of $(a+1)(b+1)(c+1)$ is $\frac{27}{8}$ 85

6997. (J. L. McKenzie, B.A.)—If Δ be the area of the triangle ABC, prove that

$$4\Delta^2 = abc(a+b+c)(1-\sin^2 \frac{1}{2}A - \sin^2 \frac{1}{2}B - \sin^2 \frac{1}{2}C); \\ \text{and } 8\Delta^3 = \frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)\sin^2(A+B)\sin^2(B+C)\sin^2(C+A)}{\sin(A-B)\sin(B-C)\sin(C-A)}.$$

..... 113

6998. (W. R. Westropp Roberts, M.A.) — Determine two independent conditions that the line, the coordinates of which are a , b , c , f , g , h , should lie in the plane $\alpha x + \beta y + \gamma z + \delta w = 0$; and from these conditions show how to write down immediately the $n+1$ conditions involving the coordinates, that a line may lie in a surface of the n th degree, by writing its equation in a symbolical form. 91

7002. (Professor Sylvester, F.R.S.)—Let s be any series of positive integers beginning with 0 and ending with n , subject to the condition that no term shall exceed the mean between the term which precedes and the term which follows it; show that the number of such series s is equal to the number of indefinite partitions of n 101

7005. (Professor Wolstenholme, M.A.)—Show in one figure the curves

- (1) $x^2 + ay = a^2$, (2) $y^2 + ay + ax = a^2$, (3) $x^2 - y^2 = ax$,
 (4) $y^2 - xy = a^2$, (5) $x^2 + y^2 + ax + 2ay = 2a^2$,
 (6) $x^2 - xy + y^2 + ay = 2a^2$, (7) $xy + ax + ay = 0$,
 (8) $x^3 + y^3 + 4axy + a^3 = 0$: which have in common the points

(x_1, y_1) , (x_2, y_2) , (x_3, y_3) , where

$$x_1 = -.4450418a, \quad x_2 = 1.2469796a, \quad x_3 = -1.8019376a,$$

$$y_1 = .8019376a, \quad y_2 = -.5549582a, \quad y_3 = -2.2469796a.$$

These points satisfy all the following equations:—

$$x_2 + y_3 = x_3 + y_1 = x_1 + y_2 = -a, \quad x_2y_2 = x_1y_3 = x_3y_1 = a^2,$$

$$x_2x_3 + x_3x_1 + x_1x_2 = -2a^2, \quad x_1x_2x_3 = y_1y_2y_3 = a^3,$$

$$x_1 + x_2 + x_3 = -a, \quad (a + x_1)(a + x_2)(a + x_3) = -a^3,$$

$$y_1 + y_2 + y_3 = -2a, \quad (a + y_1)(a + y_2)(a + y_3) = -a^3, \text{ \&c. \&c.... } 97$$

7007. (Professor Hudson, M.A.)—An endless elastic ring, natural length $2\pi l$, rests in a horizontal circle on a rough sphere, radius a , coefficient of friction $\tan \gamma$, and is just on the point of slipping when stretched to double its natural length: prove that, if the weight be neglected, $2l = a \cos \gamma$; and if the weight be w , $w \tan(a + \gamma) = 2\pi\lambda$, where α is the angular radius of the small circle formed by the string, and λ is the modulus of elasticity.

..... 115

7008. (Professor Malet, M.A.)—If the solutions of the linear differential equations $\frac{d^2y}{dx^2} + Py = 0$, $\frac{d^2y}{dx^2} + Qy = 0$, be respectively $y = C_1y_1 + C_2y_2$, $y = C_3y_3 + C_4y_4$, where y_1, y_2, y_3, y_4 are functions of x , and C_1, C_2, C_3, C_4 arbitrary constants; prove that, Δ being a constant,

$$\Delta \equiv \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} & \frac{dy_4}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} & \frac{d^2y_4}{dx^2} \\ \frac{d^3y_1}{dx^3} & \frac{d^3y_2}{dx^3} & \frac{d^3y_3}{dx^3} & \frac{d^3y_4}{dx^3} \end{vmatrix} = \Delta (P - Q)^2. \dots\dots 119$$

7009. (The Editor.)—See 6176..... 92

7010. (J. J. Walker, M.A.)—Referring to the Editor's Question 6984, show that, in the spherical triangle, the sines of the segments into which the arc joining the middle point of base with the centre of the inscribed (or escribed) circle divides the arc joining the vertex with the point of contact of that circle with the base, are as the cosines of the half sum of sides and of half the base. 96

7013. (G. F. Walker, M.A.) — Show that the problem of drawing through a given point within a given parabola a chord of the parabola which shall be divided in a given ratio at that point, has only two solutions, although at first sight it appears to have four. If POQ , $P'OQ'$ be two chords passing through O and divided in the same given ratio at that point, and if the tangents at P and P' meet in T and the tangents at Q and Q' in T' ; then TT' is parallel to the axis, and passes through O . If $T'TO$ meet the curve in R , then $RO^2 = RT \cdot RT'$ 116

7020. (R. Tucker, M.A.) — Prove that the envelop of the auxiliary circles of a system of ellipses, of equal major axes, whose foci lie on two fixed straight lines which intersect in a point on the system, is two parallel straight lines. 93

7024. (J. Macleod, M.A.) — In the produced part of the diameter of a circle, find a point from which, if a tangent be drawn to the circle, it shall be n times the produced part of the diameter, n being any number, whole or fractional, greater than unity. 96

7028. (Professor Townsend, F.R.S.) — Assuming Dr. C. TAYLOR'S theorem, that, for a curve of class p , the locus of the intersections of orthogonal tangents is of degree $p(p-1)$; show that, for two curves of classes p and q , the locus of the intersections of tangents to either with orthogonal tangents to the other is of degree $2pq$ 112

7029. (Professor Malet, M.A.) — If through a focus of an ellipsoid of revolution any three mutually rectangular chords be drawn, prove that the sum of their reciprocals is constant. 108

7042. (J. Griffiths, M.A.) — If m and n are positive integers ($m > n$), show that
$$\frac{2^{-n}}{n!} + \frac{(m-n)(m-n-1)}{(n+1)!} \cdot 2^{-(n+2)}$$
$$+ \frac{1}{2!} \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{(n+2)!} \cdot 2^{-(n+4)} + \dots$$
$$= \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2m-1)}{(m+1)!} \dots \dots \dots 112$$

7046. (S. Constable.) — Prove that the three straight lines each passing through the centre of an escribed circle of a triangle, and through the middle point of the corresponding base, pass through a common point. 113

7049. (T. Woodcock, B.A.) — Prove that

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \dots \text{ad. inf.} = \frac{1}{2\sqrt{2}} \left\{ \frac{1}{2}\pi + \log(\sqrt{2}-1) \right\}. \dots \dots 109$$

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6256. (By the Rev. C. TAYLOR, D.D.)—Determine the envelop of the circle through the three points in which a triad of concurrent lines drawn from the vertices of a given triangle meet the opposite sides.

6813. (By W. J. C. SHARP, M.A.)—If a conic pass through the three points in which a triad of concurrent lines from the vertices of a triangle meet the opposite sides, the lines joining its other points of intersection with the sides to the opposite angles will be concurrent. If

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

be the equation to such a conic referred to the triangle as triangle of reference, $ax^2 + by^2 + cz^2 - 2fyz - 2gzx - 2hxy = 0$ breaks up into straight lines.

Solution by W. J. C. SHARP, M.A.; Prof. GENESE, M.A.; and others.

(6256). The system of circles does not admit of an envelop of the ordinary description, since the equation to the circle involves three independent conditions, and only one is given. If, however, Dr. TAYLOR means an envelop of the kind mentioned in SALMON's *Higher Plane Curves* (p. 71), where the equation involves two or more independent parameters, it can doubtless be found, but the working would be very long.

(6813.) In TOWNSEND's *Modern Geometry*, Vol. I., p. 183, it is shown that, if a circle pass through the intersections with the opposite sides of three concurrent lines from the vertices of a triangle, the lines joining the other points of intersection of the circle, and the sides to the opposite angles, will also be concurrent; and the same proof, or the method of projection, will show that the property holds for all conics.

If
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

be such a conic, referred to the triangle as triangle of reference, and if (α, β, γ) be one of the points in which the connectors meet,

$$a\alpha^2 + 2h\alpha\beta + b\beta^2 = 0, \quad b\beta^2 + 2f\beta\gamma + c\gamma^2 = 0, \quad c\gamma^2 + 2g\gamma\alpha + a\alpha^2 = 0;$$

VOL. XXXVII.

C

therefore, by elimination, $abc - 2fgh - af^2 - bg^2 - ch^2 = 0$,
 which is the discriminant of $ax^2 + by^2 + cz^2 - 2fyz - 2gzx - 2hxy = 0$.

If the conic be a circle, its equation is

$$(\lambda x + \mu y + \nu z)(ax + by + cz) - (ayz + bzx + cxy) = 0,$$

$\lambda x + \mu y + \nu z = 0$ being its chord of intersection with the circumscribing circle of the triangle; and since, by the above condition,

$$4abc\lambda\mu\nu - (\mu c + \nu b - a)(\nu a + \lambda c - b)(\lambda b + \mu a - c) - \lambda a(\mu c + \nu b - a)^2 - \mu b(\lambda c + \nu a - b)^2 - \nu c(\lambda b + \mu a - c)^2 = 0,$$

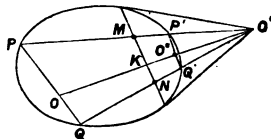
this chord envelops a class cubic of which this is the tangential equation.

The class of circles described by Dr. TAYLOR includes the nine-point circle, the two points through which the concurrent lines are drawn being the centroid and the centre of perpendiculars; and the four circles touching the sides, for each of which the two sets of concurrent lines coincide. The lines joining the points of contact to the opposite angles being concurrent in each case.

6893. (By Professor WOLSTENHOLME, M.A.)—A chord PQ of a given conic is drawn through a fixed point O; O' is another fixed point; and PP', QQ' are chords of the conic drawn through O'; prove that (1) P'Q' passes through a third fixed point O'' such that O, O', O'' are collinear; and (2) OO'' is divided harmonically by O' and the polar of O' with respect to the given conic.

Solution by CHARLOTTE A. SCOTT; J. YOUNG, B.A.; and others.

Let the polar of O' meet PP', QQ', OO' in M, N, K; and let P'Q' meet OO' in O''; then PP', QQ' are divided harmonically at M, O'; N, O'; hence OO'' is divided harmonically at K, O'; therefore, O, K, O' being fixed points, O'' is also a fixed point; therefore, &c.



5052. (By Professor SYLVESTER, F.R.S.)—Defining the *Bipotential* of an element of volume at a point-pair to mean the mass of the element multiplied by the product of the inverse distances of the element from the two points of the pair, and the bipotential of a volume to mean the sum of the bipotentials of its elements, prove that (1) the bipotential of a given uniform spherical shell in respect to an external point-pair is a function only of the bipotential at that point-pair of a unit element at the centre and the angle which the line joining the two points subtends at the centre; (2) the same proposition is true for the bipotential of the shell in respect to a point-pair lying entirely within the shell, but not for a point-pair of which one point lies within and the other outside the shell; (3) if a plane bipo-

tential is defined as above, save that the product of the logarithms of the two distances is substituted in place of the product of their inverses, then the difference between the bipotential of a circular annulus at any external point-pair and of the annulus concentrated at its centre at the same point-pair depends only on the product of the distances of the two points from the centre and the angle which the line joining them subtends at the centre.

Solution by W. J. C. SHARP, M.A.

With the usual notation, if c be the radius of the shell, k the mass of the unit of area, d and d' the distances of the points from the centre, the bipotential (1) is

$$k \int_0^\pi \int_0^{2\pi} \frac{c^2 \sin \theta \, d\theta \, d\phi}{\{c^2 - 2cd\mu + d^2\}^{\frac{1}{2}} \{c^2 - 2cd'\mu' + d'^2\}^{\frac{1}{2}}},$$

[when

$$\mu = \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos (\phi - \beta),$$

$$\mu' = \cos \theta \cos \alpha' + \sin \theta \sin \alpha' \cos (\phi - \beta'),$$

$(\alpha, \beta), (\alpha', \beta')$ being the values of α and ϕ at the points].

$$\begin{aligned} \text{Now} \quad & \int_0^\pi \int_0^{2\pi} \frac{\sin \theta \, d\theta \, d\phi}{\{1 - 2h\mu + h^2\}^{\frac{1}{2}} \{1 - 2h'\mu' + h'^2\}^{\frac{1}{2}}} \\ &= \int_{-1}^1 \int_0^{2\pi} \frac{d\mu \, d\phi}{\{1 - 2h\mu + h^2\}^{\frac{1}{2}} \{1 - 2h'\mu' + h'^2\}^{\frac{1}{2}}} \\ &= \int_{-1}^1 \int_0^{2\pi} \{1 + Y_1 h + Y_2 h^2 + \&c.\} \{1 + Z_1 h' + Z_2 h'^2 + \&c.\} \, d\mu \, d\phi, \end{aligned}$$

(where $Y_1, Y_2, \&c., Z_1, Z_2, \&c.$ are Laplace's coefficients)

$$= \int_{-1}^1 \int_0^{2\pi} \{1 + Y_1 Z_1 h h' + Y_2 Z_2 h^2 h'^2 + \&c.\} \, d\mu \, d\phi.$$

$$\text{Since} \quad \int_{-1}^1 \int_0^{2\pi} Y_m Z_n \, d\mu \, d\phi = 0$$

$$= 4\pi \left\{ 1 + \frac{Z_1'}{3} h h' + \frac{Z_2'}{5} h h'^2 + \&c. \right\} \text{ (Todhunter's Functions, pp. 152-8)}$$

therefore the Bipotential

$$= 4k\pi \frac{c^2}{dd'} \left\{ 1 + \frac{Z_1'}{3} \left(\frac{c^2}{dd'} \right) + \frac{Z_2'}{5} \left(\frac{c^2}{dd'} \right)^2 + \&c. \right\} \text{ (for two external points)}$$

$$= 4k\pi \left\{ 1 + \frac{Z_1'}{3} \left(\frac{dd'}{c^2} \right) + \frac{Z_2'}{5} \left(\frac{dd'}{c^2} \right)^2 + \&c. \right\} \text{ (for two internal points)}$$

$$= 4\pi k \frac{c}{d'} \left\{ 1 + \frac{Z_1'}{3} \left(\frac{d}{d'} \right) + \frac{Z_2'}{5} \left(\frac{d}{d'} \right)^2 + \&c. \right\}$$

(for two points one external and the other internal),

where $Z'_n = P_n(\cos A)$, A being the angle subtended by the points at the centre.

$$\log [1 - 2h \cos \alpha + h^2]^{\frac{1}{2}} = - \left(h \cos \alpha + \frac{h^2}{2} \cos 2\alpha + \frac{h^3}{3} \cos 3\alpha \right),$$

therefore the Bipotential [as defined in (3)]

$$= k \int_0^{2\pi} \{ \log [c^2 - 2cd \cos (\theta - \alpha) + d^2]^{\frac{1}{2}} \cdot \log [c^2 - 2cd' \cos (\theta - \alpha' + d'^2)]^{\frac{1}{2}} \} \, cd \, d\theta$$

[where (d, α) , (d', α') are the polar coordinates of the points] for an external point, therefore it

$$\begin{aligned}
 &= \int_0^{2\pi} c \left\{ \log d - \left[\frac{c}{d} \cos(\theta - \alpha) + \frac{c^2}{2d^2} \cos 2(\theta - \alpha) + \&c. \right] \right\}, \\
 &\left\{ \log d' - \left[\frac{c}{d'} \cos(\theta - \alpha') + \frac{c^2}{2d'^2} \cos 2(\theta - \alpha') + \&c. \right] \right\}, \\
 &= 2\pi \left\{ \log d \log d' + \frac{c^2}{dd'} \cos(\alpha - \alpha') + \left(\frac{c^2}{2dd'} \right)^2 \cos 2(\alpha - \alpha') \right. \\
 &\quad \left. + \left(\frac{c^2}{3dd'} \right)^2 \cos 3(\alpha - \alpha') + \&c. \right\},
 \end{aligned}$$

since $\int_0^{2\pi} \cos(n\theta + \beta) d\theta = 0$, if n is an integer.

6743. (By the Editor.)—Find the probability that the perimeter of a triangle whose corners are the centre of a given circle and two random points within the circle is less than three times the radius.

Solution by Prof. CAVALLIN, M.A.; D. EDWARDS; and others.

The sought probability is evidently the same if for one of the random points we take a fixed point in the circumference.

The favourable cases are then measured by the area which the given circle has in common with an ellipse having its foci at the fixed point and the centre of the circle, its major axis = $2a$, and its minor axis = $\frac{1}{2}\sqrt{3} \cdot a$, where a is the radius of the circle. Since the chord common to the circle and the ellipse bisects the ellipse and subtends at the centre of the circle an angle = $\frac{2}{3}\pi$, the required probability is

$$\left(\frac{\sqrt{3}}{4} \pi a^2 + \frac{1}{3} \pi a^2 - \frac{\sqrt{3}}{4} a^2 \right) + \pi a^2 = \frac{\sqrt{3}}{4} + \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.6285...$$

[The Editor's solution of another case of the problem, under Question 1137, is given on p. 76 of Vol. II. of the *Reprints*.]

6867. (By T. MUIR, M.A., F.R.S.E.)—M. LANDRY has shown that $2^{64} + 1$ is divisible by $1071 \cdot 2^8 + 1$. Prove that, as a consequence of this, the numbers $1071^2 + 16777216^2$, $1071^4 + 256^4$, $1071^8 + 1^8$ are also so divisible.

Solution by SARAH MARKS; the PROPOSER; and others.

If $(-1071)^r + 2^{64-8r} \equiv 0 \pmod{1071 \cdot 2^8 + 1}$, then

$$-1071 \{ (-1071)^r + 2^{64-8r} \} + 2^{64-8(r+1)} - 2^{64-8(r+1)} \equiv 0,$$

that is, $(-1071)^{r+1} + 2^{64-8(r+1)} - 2^{64-8(r+1)}(1071 \cdot 2^8 + 1) \equiv 0$,

therefore $(-1071)^{r+1} + 2^{64-8(r+1)} \equiv 0$.

But (datum) the first congruence holds when $r = 0$, therefore it holds when $r = 1$, therefore when $r = 2, \dots, = 4, \dots, = 8$, as was to be proved.

6908. (By SARAH MARKS.)—P is a point in any conic, PK the perpendicular therefrom on the directrix; with the usual notation PS is produced to F such that $SF = AX$; the circle through PAF cuts the axis in M: prove that KPMS is a parallelogram.

Solution by MARY S. MEYER; KATE GALE; and others.

Since A, P, M, F are on a circle,

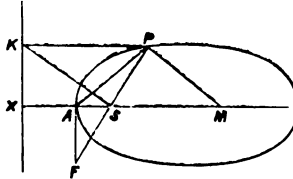
$$\angle SAF = \angle SPM,$$

and $SFA = \angle PMS$;

$$\therefore SP : SM = SA : SF (= AX) \\ = SP : SK,$$

therefore $SM = PK$;

therefore &c.



5694. (By Professor LLOYD TANNER, M.A.)—Prove that every power of n is of the form $k(1 + n + n^2 + \dots + n^r) + n'$, where n' is 1, n , n^2 , ... or n^r . For instance, every power of 3 is of one of the forms $4k \pm 1$; or it is one of the forms $13k + 1$, $13k + 3$, $13k - 4$; and so on.

Solution by H. L. ORCHARD, M.A.; E. RUTTER; and others.

Since $n^r - 1 = (n - 1)(1 + n + n^2 + \dots + n^{r-1})$,

any even power of n , n^{2r} , is

$$= n^{r+1} \times n^{r-1} = n^{r-1} \{1 + (n-1)(1 + n + n^2 + \dots + n^r)\} \\ = n^{r-1} + (n-1)n^{r-1} \{1 + n + n^2 + \dots + n^r\},$$

or $= n^{r-1} + (n-1) \{ (1 + n + n^2 + \dots + n^{r-2})(n-1) + 1 \} \{1 + n + n^2 + \dots + n^r\}$;

and any odd power, $n^{2r+1} = n^r + n^r(n-1) \{1 + n + n^2 + \dots + n^r\}$

$$= n^r + (n-1) \{1 + (n-1)(1 + n + n^2 + \dots + n^{r-1})\} \{1 + n + n^2 + \dots + n^r\}.$$

If $n = 3$ and $r = 2$, we have

$$3^4 = 3 + 2 \{2 + 1\} \{1 + 3 + 9\} = 81, \text{ of the form } 4k + 1 \text{ or } 13k + 3.$$

$$\text{So } 3^5 = 9 + 2 \{8 + 1\} \{1 + 3 + 9\} = 243, \text{ of the form } 4k - 1 \text{ or } 13k - 4.$$

6796. (By PROFESSOR CROFTON, F.R.S.)—Prove that

$$e^{DxD} \cdot e^{hx} = \frac{1}{1-h} e^{\frac{hx}{1-h}}.$$

Solution by E. BUSK, B.A.; Prof. MATZ, M.A.; and others.

$$\begin{aligned} D h^n &= n x^{n-1}, \text{ therefore } D x D (x^n) = n^2 x^{n-1}, \\ \text{therefore } (D x D)^r x^n &= n^2 (n-1)^2 \dots (n-r+1)^2 x^{n-r}, \\ \text{therefore } e^{D x D} \frac{h^n x^n}{n!} &= \left\{ 1 + \frac{E}{1!} + \frac{E^2}{2!} + \dots + \frac{E^n}{n!} \right\} \frac{h^n x^n}{n!} \\ &= \frac{h^n}{n!} \cdot \left\{ x^n + \frac{n^2}{1!} \cdot x^{n-1} + \frac{n^2 (n-1)^2}{2!} x^{n-2} + \dots + \frac{(n!)^2}{n!} \right\}. \\ \text{Similarly } e^{D x D} \frac{h^{n+1} x^{n+1}}{(n+1)!} &= \frac{h^{n+1}}{(n+1)!} \left\{ x^{n+1} + \frac{(n+1)^2}{1!} x^n + \dots \right\} \\ \therefore e^{D x D} e^{hx} &= \sum_{n=0}^{\infty} \frac{x^n h^n}{n!} + \frac{h^{n+1}}{(n+1)!} \cdot \frac{[(n+1)!]^2}{1! (n!)^2} + \frac{h^{n+2}}{(n+2)!} \cdot \frac{[(n+2)!]^2}{2! (n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{x^n h^n}{n!} \left\{ 1 + \frac{h}{1!} \cdot \frac{(n+1)!}{n!} + \frac{h^2}{2!} \cdot \frac{(n+2)!}{n!} + \dots \right\} \\ &= \sum_{n=0}^{\infty} \frac{x^n h^n}{n!} \cdot (1-h)^{-(n+1)} \\ &= \frac{1}{1-h} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{hx}{1-h} \right)^n = \frac{1}{1-h} e^{\frac{hx}{1-h}}. \end{aligned}$$

6799. (By JOHN GRIFFITHS, M.A.)—If x_1, x_2, x_3, x_4 are the roots of the equation $x^4 - \mu x^3 - \left(\frac{t^2}{e^2} + 1 - \frac{\mu^2}{4} \right) x^2 + \mu x + \frac{t^2}{e^4} - \frac{\mu^3}{4} = 0$,

where t is a variable, and μ, e constants; show (1) that

$$\int \left(\frac{1-e^2 x^2}{1-x_1^2} \right)^{\frac{1}{2}} dx_1 + \int \left(\frac{1-e^2 x_2^2}{1-x_2^2} \right)^{\frac{1}{2}} dx_2 + \dots dx_3 + \dots dx_4 = 2t + e;$$

and (2), taking $\mu < 2$ and $e < 1$, deduce FAGNANI'S theorem with respect to two elliptic arcs.

Solution by the PROPOSER; Prof. HADAMARD; and others.

1. The biquadratic may be put in the form

$$\begin{aligned} \frac{x - \frac{1}{2}\mu}{e^2} &= \left(\frac{1-e^2 x^2}{1-x_1^2} \right)^{\frac{1}{2}}; \text{ therefore } \left(\frac{1-e^2 x_1^2}{1-x_1^2} \right)^{\frac{1}{2}} dx_1 + \left(\frac{1-e^2 x_2^2}{1-x_2^2} \right)^{\frac{1}{2}} dx_2 + \&c. \\ &= \frac{x_1 dx_1 + x_2 dx_2 + \&c.}{\frac{t}{e^2}} - \frac{\frac{1}{2}\mu}{\frac{t}{e^2}} (dx_1 + dx_2 + \&c.); \end{aligned}$$

but $dx_1 + \dots + dx_4 = 0$, and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu^2 + 2 \left(\frac{t^2}{e^2} + 1 - \frac{1}{2}\mu^2 \right)$,
therefore $x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4 = \frac{2tdt}{e^2}$; therefore, &c.

2. Let ta be the line which is to be the difference between the two elliptic arcs; then, using the well-known notation,

$$\frac{(a^2 - b^2) \sin \omega \cos \omega}{p} = ta, \text{ therefore } \frac{(a^2 - b^2) x/p}{a^2 b^2} = ta,$$

therefore

$$\frac{x^2 \left(1 - \frac{x^2}{a^2} \right)}{\left\{ \frac{x^2}{a^4} + \frac{1}{b^2} \left(1 - \frac{x^2}{a^2} \right) \right\}} = \frac{t^2 a^2}{e^4}.$$

This equation gives a pair of Fagnani's points in terms of t . On reduction and making $\frac{x}{a} = x$, we get $x^4 - \left(\frac{t^2}{e^2} + 1 \right) x^2 + \frac{t^2}{e^4} = 0$, which coincides with the equation in the question on making μ zero.

If $x_1, \dots, x_4$ be the roots of this equation, we have $x_1 = -x_2$, $x_2 = -x_4$,

$$\therefore \int_0^{x_1} \frac{(1 - e^2 x_1^2)^{\frac{1}{2}}}{(1 - x_1^2)^{\frac{1}{2}}} dx_1 - \int_0^{-x_1} \frac{(1 - e^2 x_1^2)^{\frac{1}{2}}}{(1 - x_1^2)^{\frac{1}{2}}} dx_1 + \&c. = 2t + 2 \int_0^1 \frac{(1 - e^2 x^2)^{\frac{1}{2}}}{(1 - x^2)^{\frac{1}{2}}} dx,$$

therefore $2 \int_0^{x_1} \frac{(1 - e^2 x_1^2)^{\frac{1}{2}}}{(1 - x_1^2)^{\frac{1}{2}}} dx_1 - 2 \int_{x_2}^1 \frac{(1 - e^2 x_2^2)^{\frac{1}{2}}}{(1 - x_2^2)^{\frac{1}{2}}} dx_2 = 2t,$

therefore, &c.

6105. (By G. HEFFEL, M.A.)—A and B are circulating decimals, each with a recurring period of eleven figures, and $B = (A + 2)^{-1}$; find A and B.

Solution by the PROPOSER; G. EASTWOOD, M.A.; and others.

The prime factors of 9999999999 are $3^2 \times 21649 \times 513239$, and if $A + 2$ and B are represented by their equivalent fractions, the two larger factors must occur each in the denominator of the one and the numerator of the other. The only possible solution is when

$$A + 2 = \frac{513239}{3^2 \times 21649}, \quad B = \frac{3^2 \times 21649}{513239},$$

when $A = .\dot{6}341427112\dot{3}$, $B = .\dot{3}796301528\dot{1}$.

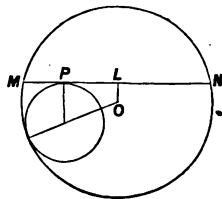
6821. (By Professor CAVALLIN, M.A.)—A point P is taken at random in the area of a vertical circle, and another point Q at random in the segment *below* the horizontal chord through P; show that the probability

that the time of descent for a particle down the straight line PQ is less than that from P down the straight line of quickest descent to the circle, is $\frac{1}{3}$.

Solution by D. EDWARDS; BELLE EASTON; and others.

Let O be the centre of the given circle, MPN the horizontal chord, OL perpendicular to MN, PL = x , OL = k , ML = l , r the radius of variable circle, A the area of the segment below the chord; then we have

$$\begin{aligned}
 p &= \int_{-a}^a \int_{-l}^l \pi r^2 dx dk + \int_{-a}^a 2l \cdot A dx \\
 &= \int_{-a}^a \int_{-l}^l \pi r^2 dx dk + \int_0^\pi 2a^4 \sin^2 \theta \times (\pi - \theta + \sin \theta \cos \theta) d\theta \\
 &= \frac{2}{\pi^2 a^4} \int_{-a}^a \int_{-l}^l \pi r^2 dx dk \\
 &= \frac{2}{\pi^2 a^4} \cdot \frac{1}{2} \cdot \pi^2 a^4 \text{ (by the solution of Quest. 6248)} \\
 &= \frac{1}{3}. \quad [\text{See Solution of Quest. 6248, Reprint, Vol. XXXIV., p. 111.}]
 \end{aligned}$$



6866. (By the Rev. A. J. C. ALLEN, B.A.)—K is the orthocentre of a triangle ABC, and a second triangle A'B'C' is formed by drawing through ABC lines parallel to the opposite sides; two conics are drawn having K as a common focus, one inscribed in the triangle ABC, the other in A'B'C'; prove that the latus rectum of one is double that of the other.

Solution by J. YOUNG, B.A.; T. WOODCOCK, B.A.; and others.

Let O be the centre of the circle circumscribing the triangle ABC; then $\angle OAB = \frac{1}{2}\pi - C = KAC$; $\angle OBC = KBA$;

hence O is the other focus of the conic inscribed in ABC. Again, K—the centre of the circle circumscribing A'B'C', and therefore the second focus of the inscribed conic—must be the orthocentre of A'B'C'. Thus the two conics have two pairs of corresponding points for foci, and are therefore similar; the triangles being similar also, every line connected with A'B'C' and its inscribed conic is double the corresponding line of ABC.

[The Question is the reciprocal with regard to K of the theorem that, if A'B'C' be the pedal triangle of ABC, the radius of the circle ABC is double that of A'B'C'.]

6875. (By R. TUCKER, M.A.)—The sides of a triangle are in arithmetical progression, and the cosine of the mean angle = $\frac{1}{\sqrt{3}}$: (1) find the sides; and (2) prove that the difference between the greatest and least angles = $\cos^{-1} \frac{1}{3}$.

Solution by ARTHUR COHEN, Q.C., M.P.; R. KNOWLES, B.A.; *and others.*

Let $x-d$, x , $x+d$ represent the sides; then we have

$$\frac{(x+d)^2 + (x-d)^2 - x^2}{2(x^2 - d^2)} = \frac{9}{16},$$

whence $x = 5d$, and the sides would be proportional to $4x$, $5x$, $6x$;
therefore $\cos(A-C) = \cos A \cos C + \sin A \sin C$

$$= \frac{1}{2} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{2} \cdot 7 = \frac{11}{3} = \frac{4}{3}.$$

6652. (By J. HAMMOND, M.A.)—Show that

$$\frac{1}{m} + \int_0^1 \int_0^1 e^{xy} (1-y)^{m-1} x^m dx dy = \int_0^1 e^x (1-x)^{m-1} dx.$$

Solution by D. EDWARDES; Rev. U. J. KNISELY, M.A.; *and others.*

More generally, if $f(x)$ be any function of x , such that the integrals

$$\begin{aligned} \text{are finite, } \int_0^1 \int_0^1 f(xy) (1-y)^{m-1} (1-x)^{n-1} x dy dx \\ = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \int_0^1 f(x) (1-x)^{m+n-1} dx \dots\dots\dots (A). \end{aligned}$$

If $f(x) = e^x$, then, integrating by parts the right-hand side, and putting $n=1$, we have the result in the Question.

[For proofs of equation (A), and of the general case for n variables, see LIOUVILLE'S *Journal de Mathématiques*, 2nd Series, Vol. IV., April 1859.]

6594. (By the EDITOR.)—A troop of Horse Artillery, on the march from Woolwich to Chatham, met a company of Marines marching at the rate of 3 miles an hour towards Woolwich; 4 miles further on, and 4 hours after leaving Woolwich, they met some stragglers from the company hastening on at the rate of 4 miles an hour, which pace would just enable them to join the company as it marched into Woolwich: find (1) at what rate the troop was marching, (2) at what distance from Woolwich it met the company of Marines, and (3) interpret the negative result.

Solution by J. O'REGAN; E. RUTTER; *and others.*

Let x denote the troop's rate of marching, in miles per hour; then $4x-4$ and $4x$ are the respective distances from Woolwich where it met the company of marines and the stragglers; also the time occupied by the troop in marching 4 miles is $\frac{4}{x}$, during which the marines marched $\frac{12}{x}$

miles towards Woolwich; hence $\frac{1}{3} \left(4x - 4 - \frac{12}{x} \right) = \frac{4x}{4}$, or $x^2 - 4x = 12$,
 an equation from which we find $x = 6$, or -2 . The troop's rate per hour
 was therefore 6 miles, and the distance from Woolwich where it met the
 company of Marines was $(4x - 4)$ or 20 miles.

[The explanation of the negative result is obtained by reversing every
 word in the question that implies the direction of march of the Artillery,
 thus "from Chatham to Woolwich," "were overtaken by," "before
 reaching."]

5919. (By S. TEBAY, B.A.)—Prove that the locus of the centre of ~~the~~
 circle that cuts from a given triangle the greatest possible area is the rec-
 tangular hyperbola $(b^2 - c^2) a^2 + (c^2 - a^2) b^2 + (a^2 - b^2) c^2 = 0$.

Solution by the PROPOSER; Prof. MATZ, M.A.; and others.

Let the intercepted chords subtend angles $= 2\theta, 2\phi, 2\psi$ at the centre;
 then, ρ being the radius of the circle, we must have

$$\rho^2 (\theta + \phi + \psi - \frac{1}{2} \sin 2\theta - \frac{1}{2} \sin 2\phi - \frac{1}{2} \sin 2\psi) \text{ a minimum};$$

also

$$\rho a \cos \theta + \rho b \cos \phi + \rho c \cos \psi = 2\Delta;$$

$$\therefore \sin^2 \theta d\theta + \sin^2 \phi d\phi + \sin^2 \psi d\psi = 0, \quad a \sin \theta d\theta + b \sin \phi d\phi + c \sin \psi d\psi = 0.$$

Multiply the second equation by λ and subtract; then, putting the
 coefficients of $d\theta, d\phi, d\psi = 0$, we have

$$\lambda = a^{-1} \sin \theta = b^{-1} \sin \phi = c^{-1} \sin \psi;$$

therefore

$$a^{-2} (\rho^2 - a^2) = b^{-2} (\rho^2 - b^2) = c^{-2} (\rho^2 - c^2).$$

Eliminate ρ^2 , then we have $(b^2 - c^2) a^2 + (c^2 - a^2) b^2 + (a^2 - b^2) c^2 = 0$,

which is a rectangular hyperbola, passing through the centres of the in-
 scribed and circumscribed circles. Putting $b^2 - c^2 = P, c^2 + a^2 = Q, a^2 - b^2 = R$,
 and $QRa^2 + RPb^2 + PQc^2 = S$, the coordinates of the centre are

$$-2S^{-1} \Delta Q R a, \quad -2S^{-1} \Delta R P b, \quad -2S^{-1} \Delta P Q c.$$

5337. (By J. J. WALKER, M.A.)—Show how to transform two binary
 cubics u, v by the same linear substitutions into $(a, b, c, d \mathcal{X} x, y)^3$, and, to
 a factor, $(a, -b, c, -d \mathcal{X} x, y)^3$ respectively, when this transformation is
 possible, viz., when the relation $\Delta \Theta'^2 - \Delta' \Theta^2 = 0$ is satisfied, the discrimi-
 nant of $u + \lambda v$ being written $(\Delta, \Theta, \Theta', \Delta' \mathcal{X} 1, \lambda)^4$. [See *Proceedings*
of the London Mathematical Society, Vol. III., p. 77.]

Solution by the PROPOSER.

For the forms to which u and v are to be reduced, $ac' - a'e = 0$ and $b'd - b'e = 0$, consequently their Jacobian (J) reduces to the form

$$Ax^4 + 6Cz^2y^2 + Ey^4;$$

so that, if S, T are its invariants, and H its Hessian, $\lambda J - H$ is a perfect square, where λ is determined by $4\lambda^3 - 8\lambda + T = 0$ (*Higher Algebra*, p. 191). The method of effecting the transformation is now obvious; e.g., let

$$u \equiv (3, -5, 2, 24\sqrt{x, y})^4, \quad v \equiv (3, 3, -25, 63\sqrt{x, y})^4,$$

$$\text{then} \quad J \equiv (48, -81, 158, -387, 1452\sqrt{x, y})^4,$$

$$S = 19200, \quad T = 257824, \quad \lambda = 14 \text{ (or } 7 \pm 3\sqrt{517}),$$

$$H \equiv (1023, -2889, 9583, -28233, 79647\sqrt{x, y})^4, \\ -14J + H = 351(x^2 - 10xy + 13y^2)^2.$$

$$X, Y = x - (5 \pm 2\sqrt{3})y, \quad 4\sqrt{3}x = (2\sqrt{3} - 5)X + (2\sqrt{3} + 5)Y, \\ 4\sqrt{3}y = -X + Y.$$

These substitutions reduce u and v to (to a factor)

$(13\sqrt{3} - 23)X^3 + (3\sqrt{3} - 11)X^2Y + (3\sqrt{3} + 11)XY^2 + (13\sqrt{3} + 23)Y^3$
and $(23\sqrt{3} - 39)X^3 - (11\sqrt{3} - 9)X^2Y - (11\sqrt{3} + 9)XY^2 + (23\sqrt{3} + 39)Y^3$
respectively; and it is easily verified that, thus transformed,

$$a : a' = -b : b' = c : c' = -d : d'.$$

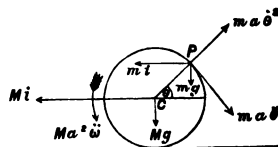
6328. (By G. J. GRIFFITHS, M.A.)—A rough uniform circular hoop, of radius a and mass M , rests in a vertical plane on a perfectly rough horizontal plane, and an insect of mass m suddenly begin to move from the extremity of a horizontal diameter, and crawls along the hoop with uniform relative velocity V ; prove that, when it arrives at the highest point of the hoop, the angular velocity ω of the hoop is given by the equation

$$(M + 2m)(V - a\omega)^2 + mga = \left(\frac{2M}{2M + m}\right)^2 (M + m) V^2.$$

Solution by D. EDWARDS; A. STEIN, M.A.; and others.

If C be the centre of the hoop, the motion of the insect is compounded of the motion of C, and of its motion relative to C. Reversing the effective forces as in the figure, and taking moments about A, the point of contact with the horizontal plane, we have

$$Ma^2 \frac{d\omega}{dt} + M\dot{v}a + m\dot{v}(a + a \sin \theta) \\ - ma^2 \dot{\theta}^2 \cos \theta - mga \cos \theta \\ - ma\ddot{\theta}(a + a \sin \theta) = 0.$$



The geometrical equations are $\dot{a}\theta = V - a\omega$, $v - a\omega = 0$, therefore

$a\ddot{\theta} = -a\dot{\omega}$. Hence the above equation becomes

$$2a^2 \left\{ M + m + m \sin \theta \right\} \dot{\omega} = m (V - a\dot{\omega})^2 \cos \theta + mga \cos \theta,$$

$$\text{or } \frac{d}{d\theta} (V - a\dot{\omega})^2 + \frac{m \cos \theta}{M + m + m \sin \theta} (V - a\dot{\omega})^2 + \frac{mga \cos \theta}{M + m + m \sin \theta} = 0.$$

$$\text{Integrating, we have } (V - a\dot{\omega})^2 = \frac{-mga \sin \theta + C}{M + m + m \sin \theta} \dots\dots\dots (1).$$

At the commencement of the motion, the external impulses upon the system act at A, and, since the whole momentum about A is zero, the initial angular velocity (ω_1) of the hoop is given by

$$2Ma^2\omega_1 = ma(V - a\omega_1), \text{ therefore } a\omega_1 = \frac{mV}{2M + m}.$$

Hence, putting $\theta = 0$ in (1), we have $C = \left(\frac{2M}{2M + m} \right)^2 (M + m) V^2$;

and therefore, putting $\theta = \frac{1}{2}\pi$, we have

$$(M + m)(V - a\dot{\omega})^2 + mga = \left(\frac{2M}{2M + m} \right)^2 (M + m) V^2.$$

6659. (By Professor TOWNSEND, F.R.S.)—In the irrotational strain of an incompressible lamina in the plane of its mass, if the potential and displacement line systems of the strain be two orthogonal systems of coaxial circles; show that, throughout the lamina, the actual molecular displacement and principal linear dilatation vary respectively, the former inversely as the product of the distances from the foci, and the latter directly as the distance from the centre and inversely as the product of the squares of the distances from the foci of the systems.

Solution by the PROPOSER.

Taking the connector of the foci and its orthogonal bisector for axes of x and of y respectively, and denoting by $2h$ the interval between the foci; then since, for any two arbitrary circles of the opposite systems having their centres at the distances a and b from the origin on the axes of x and of y respectively, the equations are respectively

$$x^2 + y^2 + h^2 = 2ax \text{ and } x^2 + y^2 - h^2 = 2by,$$

we have consequently, first to determine two functions u and v of $\frac{x^2 + y^2 + h^2}{x}$ and $\frac{x^2 + y^2 - h^2}{y}$ respectively, which will satisfy, for all values

of x and y , the two independent equations

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \text{ and } \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0,$$

as well as the two equations of connection $\frac{du}{dx} = \frac{dv}{dy}$ and $\frac{du}{dy} = -\frac{dv}{dx}$, and then deduce from them the two aforesaid particulars of the strain.

Proceeding in the usual manner, we find without difficulty, that

$$u = k \cdot \log \left[\frac{y^2 + (x \pm h)^2}{y^2 + (x \mp h)^2} \right]^{\frac{1}{2}} \text{ and } v = k \cdot \tan^{-1} \left[\frac{x^2 + y^2 - h^2}{\pm 2hy} \right],$$

where k is any constant sufficiently small to allow of the application of the ordinary laws of small quantities to the particulars of the strain; and therefore, by differentiation with respect to x and y , that

$$\left[\left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right]^{\frac{1}{2}} = \left[\left(\frac{dv}{dx} \right)^2 + \left(\frac{dv}{dy} \right)^2 \right]^{\frac{1}{2}} = \frac{2kh}{r_1 r_2},$$

where r_1 and r_2 are the distances of xy from the foci; and that

$$\left[\left(\frac{d^2u}{dx^2} \right)^2 - \frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} \right]^{\frac{1}{2}} = \left[\left(\frac{d^2v}{dx^2} \right)^2 - \frac{d^2v}{dx^2} \cdot \frac{d^2v}{dy^2} \right]^{\frac{1}{2}} = \frac{4kh}{r_1^2 r_2^2},$$

where r is the distance of xy from the centre, and r_1 and r_2 , as before, its distances from the foci; and therefore, &c., as regards both properties of the strain.

When $n = 0$, that is, when the two systems of circles pass through the origin, and there touch respectively the axes of y and x ; since then $r_1 = r_2 = r$, it follows consequently from the above that then the actual molecular displacement and principal linear dilatation of the strain vary respectively, the former inversely as the square, and the latter inversely as the cube, of the distance from the centre.

5997. (By W. J. C. SHARP, M.A.)—Prove that the catalecticant of the binary $2n$ -ic is the discriminant of an $(n+1)$ -ary quadratic; and hence deduce the properties of the catalecticant.

Solution by the PROPOSER.

The vanishing of the discriminant of the $(n+1)$ -ary quadratic is the condition that this function should be expressible as the sum of n squares.

If then the discriminant of $ax^2 + 2bxy + cy^2 + 2fyz + 2dzy + cz^2$ vanish, this function is equal to

$$A(x + \mu y + \nu z)^2 + A_1(w + \mu_1 y + \nu_1 z)^2,$$

and if $f = c$, to $A(x + \mu y + \mu^2 z)^2 + A_1(w + \mu_1 y + \mu_1^2 z)^2$;

for the requisite condition that this should be so is

$$\left\| \begin{array}{cccccc} 1 & \mu & \mu^2 & \mu & \mu^4 \\ 1 & \mu_1 & \mu_1^2 & \mu_1^3 & \mu_1^4 \\ a & b & c & d & e \end{array} \right\| = 0, \text{ or } \left| \begin{array}{ccc} a & b & c \\ b & c & d \\ c & d & e \end{array} \right| = 0,$$

a condition which already holds.

If now $x = X^2$, $y = 2XY$, $z = Y^2$, the quadratic becomes

$$(a, b, c, d, e)(XY)^4,$$

and the equivalent expression $A(X + \mu Y)^4 + A_1(X + \mu_1 Y)^4$,

whilst the discriminant is the catalecticant T .

Similarly, if the discriminant of

$$ax^2 + cy^2 + cz^2 + gu^2 + 2bxy + 2hxx + 2dxw + 2kxz + 2fzw + 2lyw,$$

vanish, this function may be put in the form

$$\Lambda(x + \lambda y + \mu z + \nu w)^2 + \Lambda_1(x + \lambda_1 y + \mu_1 z + \nu_1 w)^2 + \Lambda_2(x + \lambda_2 y + \mu_2 z + \nu_2 w)^2,$$

and if $h = c$, $k = d$, and $l = e$, in the form

$$\Lambda(x + \lambda y + \lambda^2 z + \lambda^3 w)^2 + \Lambda_1(x + \lambda_1 y + \lambda_1^2 z + \lambda_1^3 w)^2 + \Lambda_2(x + \lambda_2 y + \lambda_2^2 z + \lambda_2^3 w)^2.$$

For the necessary condition

$$\begin{vmatrix} 1 & \lambda & \lambda^2 & \lambda^3 & \dots & \lambda^6 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \dots & \lambda_1^6 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \dots & \lambda_2^6 \\ a & b & c & d & \dots & g \end{vmatrix} = 0, \text{ is equivalent to } \begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{vmatrix} \text{ the vanishing of}$$

or the discriminant. Now, if $x = X^3$, $y = 3X^2Y$, $z = 3XY^2$, $w = Y^3$,

the quadratic becomes $(a, b, c, d, e, f, g)(XY)^6$,

and its equivalent $\Lambda(X + \lambda Y)^6 + \Lambda_1(X + \lambda_1 Y)^6 + \Lambda_2(X + \lambda_2 Y)^6$,

whilst the discriminant is identical with the catalecticant of the binary quantic. A similar method of argument proves the general proposition.

6360. (By J. W. RUSSELL, M.A.)—A string AC is stretched tight, and a point B in it passes through a smooth ring; particles of equal mass attached to A and C are projected in the same plane and perpendicular to BA and to CA with equal angular momenta about B, and are acted on by no forces except the tension of the string; show that the two particles will describe the same path.

Solution by the PROPOSER; J. B. SANDERS, M.A.; and others.

The masses of the particles A and C being equal, the conservation of areas gives us the two equations $r^2\dot{\theta} = h = s^2\dot{\phi}$, r and θ being the polar coordinates of A, and s and ϕ those of B referred to the initial position of the string. The equation of energy now gives

$$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{s}^2 + s^2\dot{\phi}^2 = \text{constant}; \text{ but } r + s = 2a, \text{ the length of the string;}$$

$$\text{therefore } 2\dot{r}^2 + h^2 \left\{ \frac{1}{r^3} + \frac{1}{s^3} \right\} = \text{constant}.$$

Now, in the beginning of the motion $\dot{r} = \dot{r}_0 + \ddot{r}_0 t + \dots$; hence $\dot{r}$ has the same sign as $\ddot{r}_0$. Also, resolving along the string,

$$r\ddot{\theta}^2 - \ddot{r} = T = s\dot{\phi}^2 - \ddot{s};$$

$$\text{therefore } 2\ddot{r} = r\ddot{\theta}^2 - s\dot{\phi}^2 = h^2 \left(\frac{1}{r^3} - \frac{1}{s^3} \right), \text{ for } r + s = 2a,$$

which is negative if $r > s$ initially; therefore r decreases at first, and can only begin to increase when $\dot{r} = 0$, i.e., when $r^{-2} + s^{-2} = \text{initial value} = b^{-2} + c^{-2}$, but $r + s = b + c = 2a$. To solve these, put $(rs)^{-1} = x$;

therefore $\frac{1}{r^2} + \frac{1}{s^2} = \frac{(b+c)^2 - 2rs}{r^2 s^2} = (b+c)^2 x^2 - 2x,$

therefore $(b+c)^2 x^2 - 2x = \frac{1}{b^2} + \frac{1}{c^2},$ therefore $x = \frac{1}{bc}$ or $-\frac{b^2+c^2}{bc(b+c)}.$

Hence r vanishes only when $r = b$ or $r = c$; and r decreases from $r = b$ to $r = c$, whilst s increases from $r = c$ to $r = b$. The circumstances of A now are exactly the same as those of C initially; hence both particles describe the same path, this path having (like a central orbit) two apsidal distances.

6728. (By S. CONSTABLE, M.A.)—If O, O_1, O_2, O_3 are the centres of the inscribed and escribed circles of a triangle ABC ; $(\alpha, \beta, \gamma), (\alpha', \beta', \gamma')$ the points of contact of circles $(O), (O_1),$ &c.; $(\Delta, \Delta', \Delta''), (\delta, \delta', \delta'')$ the areas of the triangles $O_1O_2O_3, OO_2O_3,$ &c., $\alpha\beta\gamma, \alpha'\beta'\gamma',$ &c.; prove

$$\delta = \frac{S^2}{\Delta}, \quad \delta' = \frac{S^2}{\Delta'}, \quad \&c.; \quad \frac{1}{\delta'} + \frac{1}{\delta''} + \frac{1}{\delta'''} = \frac{1}{\delta}.$$

Solution by Rev. T. R. TERRY, M.A.; J. O'REGAN; and others.

$$S = 2R^2 \sin A \sin B \sin C,$$

$$\Delta = 2(2R)^2 \sin \frac{1}{2}(B+C) \sin \frac{1}{2}(C+A) \sin \frac{1}{2}(A+B) \\ = \frac{1}{2} S \operatorname{cosec} \frac{1}{2} A \operatorname{cosec} \frac{1}{2} B \operatorname{cosec} \frac{1}{2} C,$$

$$\Delta' = 2(2R)^2 \sin \frac{1}{2} B \sin \frac{1}{2} C \sin (90^\circ + \frac{1}{2} A) = \frac{1}{2} S \operatorname{cosec} \frac{1}{2} A \sec \frac{1}{2} B \sec \frac{1}{2} C, \&c.,$$

$$\delta = 2r^2 \sin \frac{1}{2}(B+C) \sin \frac{1}{2}(C+A) \sin \frac{1}{2}(A+B) \\ = 2S \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C,$$

$$\delta' = 2r'^2 \sin \frac{1}{2} B \sin \frac{1}{2} C \sin (90^\circ + \frac{1}{2} A) = 2S \sin \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C, \&c.;$$

therefore $\delta = \frac{S^2}{\Delta}, \quad \delta' = \frac{S^2}{\Delta'}, \quad \&c.,$

and obviously from the figure $\Delta = \Delta' + \Delta'' + \Delta''',$

therefore $\frac{1}{\delta'} + \frac{1}{\delta''} + \frac{1}{\delta'''} = \frac{1}{\delta}.$

6434. (By R. KNOWLES, B.A., L.C.P.)—If through a point P in a conic a tangent be drawn at right angles to a chord OQ, prove that (1) the chords OP, QP will make angles with the normal at P, such that the product of their tangents is constant; and (2) if the conic be an equilateral hyperbola, OP, QP will be at right angles.

Solution by W. H. LOWRY, M.A.; G. F. WALKER, M.A.; and others.

We have $\tan \theta \tan \theta' = \frac{ON \cdot NQ}{NP^2}$

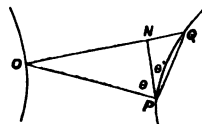
= ratio of squares of parallel diameters,
which is constant if P be fixed.

In the rect. hyperbola the ratio is -1, and

$$\tan \theta \tan \theta' = -1,$$

and therefore they are at right angles.

[See Quest. 6297, *Reprint*, Vol. XXXIII., p. 98.]



6578. (By G. F. WALKER, M.A.)—A fine circular tube, of uniform bore and of radius a , in a vertical plane, contains columns of n fluids or densities $\rho_1, \rho_2 \dots \rho_n$ respectively, the respective columns subtending angles $2a_1, 2a_2 \dots 2a_n$ at the centre of the circle. Show that, if θ be the angle, the radius to the free surface of the fluid of density ρ_1 makes with the vertical when the system is in equilibrium under the action of gravity,

$$\tan \theta = \sum_{r=1}^{r=n} \rho_r \sin a_r \sin (2s_r - a_r) + \sum_{r=1}^{r=n} \rho_r \sin a_r \cos (2s_r - a_r)$$

where s_r denotes $a_1 + a_2 + \dots + a_r$.

Solution by the PROPOSER; A. HERBERT, M.A.; and others.

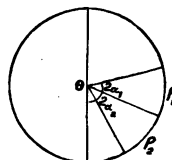
The pressures at the free ends are zero, and therefore the variation in pressure in going along the tube from one end to the other is zero. Therefore

$$\sum_{r=1}^{r=n} a g \rho_r [\cos (\theta - 2a_1 - \dots - 2a_r) - \cos (\theta - 2a_1 - \dots - 2a_{r-1})] = 0,$$

therefore $\sum_{r=1}^{r=n} \rho_r \sin a_r \sin (\theta - 2s_r + a_r) = 0,$

$$\sin \theta \sum_{r=1}^{r=n} \rho_r \sin a_r \cos (2s_r - a_r) - \cos \theta \sum_{r=1}^{r=n} \rho_r \sin a_r \sin (2s_r - a_r) = 0,$$

and $\tan \theta = \sum_{r=1}^{r=n} \rho_r \sin a_r \sin (2s_r - a_r) + \sum_{r=1}^{r=n} \rho_r \sin a_r \cos (2s_r - a_r).$



6160. (By Professor WOLSTENHOLME, M.A.)—Two fixed conjugate semi-diameters of an ellipse are CA, CB; P is any other point on the ellipse; A', B', P' are the corresponding points on the auxiliary circle; and A'a, B'b, P'p are chords of the auxiliary circle perpendicular to the major axis. Straight lines PL, PM are drawn parallel to CA', CB', and terminated by CA, CB; and straight lines Pl, Pm parallel to Ca, Cb, and also terminated by CA,

CB respectively; prove that LM, *lm* are of constant lengths, and are perpendicular respectively to CP', Cp. Their lengths are $(a \pm b) \tan ACB$.

Solution by the PROPOSER.

If Cx, Cy be two fixed straight lines, LM a constant length (c) intercepted between them, LP, MP drawn in two given directions at right angles, $xLP = \alpha$, $yMP = \beta$; therefore $\alpha + \beta = xCy + \frac{1}{2}\pi \equiv \omega + \frac{1}{2}\pi$; and if CL = p , CM = q , the equation of LP is

$$y = \frac{\sin \alpha}{\sin (\omega - \alpha)} (x - p) = -\frac{\sin \alpha}{\cos \beta} (x - p), \text{ or } p = x + \frac{y \cos \beta}{\sin \alpha};$$

$$\text{so } q = y + \frac{x \cos \alpha}{\sin \beta}; \text{ and } c^2 = p^2 + q^2 - 2pq \cos \omega,$$

$$\text{or } c^2 = p^2 + q^2 - 2pq \sin (\alpha + \beta),$$

and the equation of the locus of P is

$$c^2 = \left(x + \frac{y \cos \beta}{\sin \alpha}\right)^2 + \left(y + \frac{x \cos \alpha}{\sin \beta}\right)^2 - 2 \sin (\alpha + \beta) \left(x + \frac{y \cos \beta}{\sin \alpha}\right) \left(y + \frac{x \cos \alpha}{\sin \beta}\right),$$

and the coefficient of $2xy$ is

$$\frac{1}{\sin \alpha \sin \beta} [\sin \beta \cos \beta + \sin \alpha \cos \alpha - \sin (\alpha + \beta) \cos (\alpha - \beta)],$$

and therefore vanishes. The coefficient of x^2 is

$$\frac{\sin^2 \beta + \cos^2 \alpha - 2 \cos \alpha \sin \beta \sin (\alpha + \beta)}{\sin^2 \beta},$$

$$\text{or } \frac{\cos^2 \alpha + \sin \beta [\sin (\alpha + \beta - \alpha) - 2 \cos \alpha \sin (\alpha + \beta)]}{\sin^2 \beta},$$

$$\text{or } \frac{\cos^2 \alpha - \sin \beta \sin (2\alpha + \beta)}{\sin^2 \beta}; \text{ i.e., } \frac{1 + \cos 2\alpha - \cos 2\alpha + \cos 2(\alpha + \beta)}{2 \sin^2 \beta},$$

$$\text{or } \frac{\cos^2 (\alpha + \beta)}{\sin^2 \beta}. \text{ Hence the equation is } \frac{c^2}{\cos^2 (\alpha + \beta)} = \frac{x^2}{\sin^2 \beta} + \frac{y^2}{\sin^2 \alpha}.$$

Hence, if $2a$, $2b$ be the semi-axes, we have

$$ab = \pm c^2 \frac{\sin \alpha \sin \beta}{\cos^2 (\alpha + \beta)} \sin \omega = \mp c^2 \frac{\sin \alpha \sin \beta \cos (\alpha + \beta)}{\cos^2 (\alpha + \beta)},$$

$$\text{and } a^2 + b^2 = c^2 \frac{\sin^2 \alpha + \sin^2 \beta}{\cos^2 (\alpha + \beta)},$$

$$\text{and } (a \pm b)^2 = \frac{c^2 [\sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \cos (\alpha + \beta)]}{\cos^2 (\alpha + \beta)} = c^2 \tan^2 (\alpha + \beta),$$

$$\text{or } a \pm b = \pm c \tan (\alpha + \beta) = c \cot \omega, \text{ or } c = (a \pm b) \tan \omega;$$

determining the intercepted length. The angles are given by

$$a' = \pm \frac{c \sin \beta}{\cos (\alpha + \beta)} = \frac{a \pm b}{\cos \omega} \sin \beta; \quad b' = \frac{a \pm b}{\cos \omega} \sin \alpha,$$

where a' , b' are semi-diameters along the coordinate axes.

6602. (By Professor NASH, M.A.)—Prove that (1) the radical axis of the circles of curvature at the extremities of any ordinate to a given diameter

of a parabola passes through a fixed point, and (2) the locus of this point for different diameters is a parabola.

Solution by G. F. HOPKINS, M.A.; G. F. WALKER, M.A.; and others.

Let m_1, m_2 be the extremities of the ordinate, and m that of the diameter, then $\frac{1}{m_1} + \frac{1}{m_2} = \frac{2}{m}$. The circle of curvature at the point m is easily found

$$\text{to be } x^2 + y^2 - \left(4 + \frac{6}{m^2}\right)ax + \frac{4a}{m^2}y - \frac{3a^2}{m^4} = 0.$$

The equation to the radical axis will therefore be

$$\begin{aligned} & -6ax \left(\frac{1}{m_1^2} - \frac{1}{m_2^2} \right) + 4ay \left(\frac{1}{m_1^3} - \frac{1}{m_2^3} \right) = 3a^2 \left(\frac{1}{m_1^4} - \frac{1}{m_2^4} \right), \\ \text{or } & -6ax \left(\frac{1}{m_1} + \frac{1}{m_2} \right) + 4ay \left(\frac{1}{m_1^2} + \frac{1}{m_1 m_2} + \frac{1}{m_2^2} \right) = 3a^2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right), \\ \text{or } & -\frac{12ax}{m} + 4ay \left(\frac{4}{m^2} - \frac{1}{m_1 m_2} \right) = \frac{6a^2}{m} \left(\frac{4}{m^2} - \frac{2}{m_1 m_2} \right), \end{aligned}$$

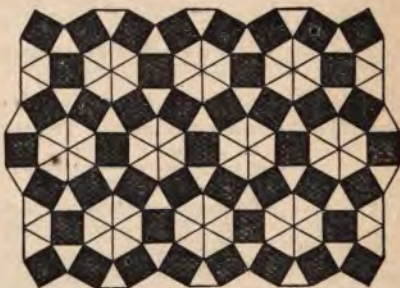
which involves the single parameter $\frac{1}{m_1 m_2}$ in the first degree, and passes through the fixed point $x = \frac{2a}{m^2}, y = \frac{3a}{m}$, whose locus is the parabola $2y^2 = 9ax$.

6750. (By the Rev. W. A. WHITWORTH, M.A.)—A pavement is made of black and white tiles, the black being squares, and the white equilateral triangles. Construct the pattern so that the areas of black and white may be in the ratio of $\sqrt{3} : 4$.

Solution by the PROPOSER.

Construct the pattern as follows:—Put 6 of the triangles together to make a hexagon; on each face of the hexagon put a square; between these 6 squares put 6 triangles; on the exposed faces of these triangles put squares; between these 6 squares put 6 hexagons as before; continue the pattern round these hexagons as round the first hexagon; and so *ad infinitum*.

It will be seen that, when the pattern is extended, 3 squares are used for every eight triangles. Hence the black and white areas will be as $3 : 4\sqrt{3}$ or as $\sqrt{3} : 4$.



6369. (By J. L. MCKENZIE, B.A.)—If $s_r = a_1^r + a_2^r + \dots + a_n^r$, where r is any positive integer, prove (1) that

$$\begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ s_{m-1} & s_{m-2} & \dots & m-1 & \\ s_m & s_{m-1} & \dots & s_1 & \end{vmatrix} = (-1)^n a_1 a_2 a_3 \dots a_n n!, \text{ when } m = n; \text{ and (2) that the determinant vanishes for any value of } m \text{ greater than } n.$$

Solution by W. J. C. SHARP, M.A. ; Prof. EVANS, M.A. ; and others.

1. If $x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots + (-1)^n p_n = 0$ be the equation whose roots are $a_1, a_2 \dots a_n$, then $s_1 - p_1 = 0$, $s_2 - p_1 s_1 + 2p_2 = 0$, &c. &c., $s_r - p_1 s_{r-1} + p_2 s_{r-2} - \dots + (-1)^r p_r = 0$; from which it follows that, if A_m denote the given determinant

$$A_n = \frac{m p_m}{p_{m-1}} A_{m-1} \text{ and } A_1 = p_1, \text{ therefore } A_m = p_m \cdot m!$$

2. Hence $A_n = p_n \cdot n! = (-1)^n a_1 a_2 a_3 \dots a_n \cdot n!$ and $A_{n+r} = 0$.

4470. (By the EDITOR.)—A heavy uniform beam (AB) moves freely about a hinge at A, and an elastic string is attached to the extremity B, and fixed at a point C in the same horizontal line as A, at a distance (AC) equal to the length of the beam. The natural length of the elastic string is equal to half that of the beam, and its elasticity is such that the weight of the beam would stretch it to twice its natural length. Find the angle which the string makes with the horizon when the system is in equilibrium.

Solution by G. S. CARR, B.A. ; Prof. EVANS, M.A. ; and others.

Let W = the weight of the beam, T = the tension of the string, 2θ = the angle required; then, by the law of elasticity, we have

$$\begin{aligned} \frac{T}{W} &= \frac{2BC}{AB} - 1 = 4 \sin \theta - 1, \text{ by the geometry of the figure,} \\ &= \frac{\cos 2\theta}{2 \cos \theta}, \text{ by taking moments about the hinge,} \end{aligned}$$

hence $\frac{\cos 2\theta}{2 \cos \theta} = 4 \sin \theta - 1$, from which $\cos \theta$ is obtainable by a biquadratic.

5951. (By D. EDWARDS.)—If A, B, C are the angles of a triangle, and if $\alpha = \frac{\sin 2B + \sin 2C}{2 \cos 2A - 1}$, with similar expressions for β and γ , prove that $\alpha + \beta + \gamma = 4\alpha\beta\gamma$.

Solution by LIZZIE A. KITTUDGE; J. O'REGAN; and others.

Multiplying out the left-hand side, the numerator is

$$\begin{aligned} & 2 \sin 2B (2 \cos 2B - 1)(\cos 2C + \cos 2A - 1) + \dots + \dots, \text{ which} = \\ & 2 \sin 4B(\cos 2C + \cos 2A) - 2 \sin 4B + 2 \sin 2B - 2 \sin 2B(\cos 2C + \cos 2A). \text{ \&c.} \\ \text{Now } & \Sigma 2 \sin 4B (\cos 2C + \cos 2A) = \Sigma 4 \sin 2A (\sin^2 2B + \sin^2 2C - 1), \\ & \Sigma \sin 4B = -4 \sin 2A \sin 2B \sin 2C, \\ & \Sigma 2 \sin 2B (\cos 2C + \cos 2A) = -\Sigma 2 \sin 2A; \end{aligned}$$

hence the numerator becomes

$$\begin{aligned} & 4 \sin 2A (\sin^2 2B + \sin^2 2C) + \dots - 4 \Sigma \sin 2A \\ & \quad + 8 \sin 2A \sin 2B \sin 2C + 2 \Sigma \sin 2A + 2 \Sigma \sin 2. \\ & = 4 \{ \sin 2A (\sin^2 2B + \sin^2 2C) + \dots + \dots + 2 \sin 2A \sin 2B \sin 2C \} \\ & = 4 (\sin 2A + \sin 2B)(\sin 2B + \sin 2C)(\sin 2C + \sin 2A). \end{aligned}$$

6929. (By Dr. DONALD McALISTER.)—Of 10 cases treated by Lister's method, 7 did well and 3 suffered from blood-poisoning; of 14 cases treated with ordinary dressings, 9 did well and 5 had blood-poisoning; what are the odds that the success of Lister's method was due to chance?

Solutions by (1) Dr. MACFARLANE, F.R.S.E.; (2) the PROPOSER.

(1) Let p denote the chance of a case treated by LISTER's method doing well, and q the chance of a case treated with ordinary dressings doing well, then $p = \frac{7}{10}$ and $q = \frac{9}{14}$. But LISTER's method consists in the ordinary dressings with the additional use of an antiseptic; hence the effect of the antiseptic is $p - q$, that is $\frac{3}{35}$. Hence the odds that in a given case the success of LISTER's method is not due to the characteristic part of it is

$$\frac{q}{p - q}, \text{ that is } \frac{45}{4}.$$

(2) I have been permitted to see the above solution, and cannot help saying that I think it very inadequate. It is a good rule in Probabilities to refrain from introducing any datum of your own into the conditions of the question. Here, however, the solver has drawn upon his knowledge of surgery, not to the advantage of his mathematics. Lister's method does not consist "in the ordinary dressings, with the additional use of an antiseptic." The only sense in which that statement could be true, would be that in which any one thing may be said to be the same as another—with the addition of the difference. But it is hard to see the force of the statement when the things differ in kind.

What is meant, then, by the phrase, "the effect of the antiseptic is $p - q$," I am wholly unable to guess. Perhaps the solver may be so good as to expand this statement into ordinary terms; at present it seems a little too condensed. I might further point out, that the solution somehow arrived at in the last sentence (I do not pretend to understand the force of the "Hence") is no answer to my question. I said nothing about "a given case," and nothing about the "characteristic

part" of the treatment applied to it. In the cases treated by the two methods, Listerism had the best of it, what is the *a posteriori* probability that Listerism has something in it which determined this result; or, on the other hand, that the result was merely due to chance? If I had set the question in a familiar form, I suspect I should not have received quite the same answer. Suppose, for example, two urns A and B, each containing a large number of black and white balls. I draw $a+b$ balls in succession from A, and of these a are white; I draw $p+q$ balls in succession from B, and of these p are white. I note that $\frac{a}{a+b}$ is greater than $\frac{p}{p+q}$. What are the odds that the proportion of white balls in B is

really less than the proportion of white balls in A? In other words, what are the odds that the apparent preponderance of white balls in A over those in B is not due to chance? The answer, obtained by means of the ordinary principles of Inverse Probabilities, may be given in this form: Let P denote the probability that there *are* relatively fewer white balls in B than in A; then $1-P$ is the probability that there is relatively (to the whole number contained) a greater or an equal number of white balls in B. Also, let n be the total number of balls in each urn; then

$$\frac{P}{1-P} = \frac{\sum_{s=1}^{n-1} \sum_{t=1}^{n-s-1} s^a (n-s)^b t^p (n-t)^q}{\sum_{s=1}^{n-1} \sum_{t=s}^{n-1} s^a (n-s)^b t^p (n-t)^q};$$

an expression giving the odds against a real preponderance of white balls in B.

Written in another form, we may give the results thus:—

$$\begin{aligned} P &= \frac{(a+b+1)!(p+q+1)!(a+p+1)!(b+q+1)!}{a!(b+1)!(p+1)!q!(a+b+p+q+2)!} \times S, \\ S &= 1 + \frac{a \cdot q}{b+2 \cdot p+2} + \frac{a \cdot a-1 \cdot q \cdot q-1}{b+2 \cdot b+3 \cdot p+2 \cdot p+3} + \&c...., \\ 1-P &= \frac{(a+b+1)!(p+q+1)!(a+p+1)!(b+q+1)!}{(a+1)!b!p!(q+1)!(a+b+p+q+2)!} \times S', \\ S' &= 1 + \frac{b \cdot p}{a+2 \cdot q+2} + \frac{b \cdot b-1 \cdot p \cdot p-1}{a+2 \cdot a+3 \cdot q+2 \cdot q+3} + \&c.... \end{aligned}$$

This has the advantage of not bringing in n (which has *not* been assumed infinite); for the series S' terminate at the first term that vanishes.

The analogy with my original problem is clear. The large number of balls in A represents the total number of observations on patients treated by Lister's method which have been recorded or may in future be recorded. The number in B may represent the total number of observations made under some other system. Now, if in the vast A series the proportion of successes were greater than in the equally vast B series, it would be granted that an advantage lay in Lister's method compared with the other. This is, in fact, our only means of testing the relative values of different methods; by their fruits we know them. Our problem is—from our small number of observed cases to gather what we may as to the probable outcome of the two methods if they were applied on the largest possible scale. It is to learn what we can from a basket of apples concerning the whole contents of the orchard.

The authorities on Probability give formulæ which only apply to "samples" of great magnitude; a mere basketful is nothing to them.

But it may fairly be urged that even a basketful gives *some* information. A buyer would not be held unwise who, knowing nothing but what the samples told him, should prefer to invest in the orchard whence came the better basket. He might reasonably ask the mathematician to appraise the amount of information the samples gave him, even though the samples were not very large. Here POISSON'S expression would be inapplicable, and the mathematician would have to go back to first principles as I have.

The answer arrived at in Dr. MACFARLANE'S solution strikes an unsophisticated mind as odd. The numbers given in the question show a preponderance of success in Lister's favour. *Knowing nothing more*, would not the non-mathematical patient prefer to take his chance under Lister's system? And could the mathematician, even though convinced that the odds were more than 11 to 1 against there being "anything in Listerism," regard his preference as unjustified? My result is very different; using the formula as given above, and taking $a = 5$, $b = 9$, $p = 3$, $q = 7$, I find,

$$1 - P = \frac{15! 11! 9! 17!}{6! 9! 3! 8! 26!} \left(1 + \frac{3 \cdot 9}{7 \cdot 9} + \frac{3 \cdot 2 \cdot 9 \cdot 8}{7 \cdot 8 \cdot 9 \cdot 10} + \frac{3 \cdot 2 \cdot 1 \cdot 9 \cdot 8 \cdot 7}{7 \cdot 8 \cdot 9 \cdot 9 \cdot 10 \cdot 11} \right).$$

Calculating this by means of DÆGEN'S table of logarithms of factorials, I

find $1 - P = \cdot 40175$, $\therefore P = \cdot 59825$, $\therefore \frac{P}{1 - P} = 1 \cdot 49$ nearly.

Thus we may wager nearly 3 to 2 that the difference in the results of the two methods of treatment is not due to chance, but (other constant disturbing causes being excluded) depends on a difference residing in the methods themselves.

Prof. LIEBERMEISTER, of Tübingen, has published several tracts in which problems similar to mine are very clearly discussed.

6919. (By Professor SYLVESTER, F.R.S.)—If $u_{x+1} - u_x^2 + u_x - 1 = 0$, prove that $\frac{1}{u_x} = \frac{1}{u_0 - 1} - \frac{1}{u_{x+1} - 1}$. Thus, suppose $u_0 = 5$; then $u_1 = 21$, $u_2 = 421$; and, if $x = 1$, we ought to find $\frac{1}{21} + \frac{1}{421} = \frac{1}{4} - \frac{1}{420}$, which is right.

[Prof. SYLVESTER states that this question arises naturally out of the very beautiful Ancient Egyptian method of expressing all fractions under the form of a sum of simple fractions with continually increasing denominators, that is, as the sum of the reciprocals of continually increasing integers. It is easy to prove that in the limiting case these integers must be subject to the law expressed by the above equation in differences. The successive integers are, as in the parallel method of continued fractions, supposed to be all made maxima, which renders the process, in this as for that method, perfectly determinate; though the Egyptians did not uniformly observe this condition in their praxis. The successive fractions, it is easy to conclude from the above equation, (as an example taken at random, such as $\sqrt{2}$ for instance, will readily confirm,) decrease with enormous rapidity. The Proposer adds, that this easy but deeply interesting question—which he wishes to bear the title of *Egyptian Arithmetic*—furnishes a new instance of the wonderful wisdom of the Egyptians.]

Solution by Prof. TANNER, M.A.; Rev. R. HARLEY, F.R.S.; and others.

The given equation may be written $\frac{1}{u_n} = \frac{1}{u_n - 1} - \frac{1}{u_{n+1} - 1}$, which leads at once to

$$\sum \frac{1}{u_n} = \frac{1}{u_0 - 1} - \frac{1}{u_{n+1} - 1}.$$

In reference to the interesting note on Egyptian Arithmetic, it may be worth while to remark that the denominators of the successive fractions are the "quotients" in the following G. C. M.-like process. The first divisor and dividend are the numerator and denominator of the proposed fraction. Each quotient $\times$ corresponding divisor gives a product *not less* than the dividend. The excess serves for the next divisor; and every dividend is the product of quotient and dividend of the preceding stage.

If each quotient had been found as in ordinary division, the rest of the process being unaltered, we should get the denominators of a series of fractions, the numerators being ± 1 alternately, the sum of which is again equal to the original fraction. For examples take the following:—

$$\begin{array}{r} 5 \overline{) 17} \mid 4 \\ \underline{20} \\ 3 \overline{) 68} \mid 23 \\ \underline{69} \\ 1 \overline{) 1564} \mid 1564 \\ \underline{1564} \end{array}$$

$$\begin{array}{r} 5 \overline{) 17} \mid 3 \\ \underline{15} \\ 2 \overline{) 51} \mid 25 \\ \underline{50} \\ 1 \overline{) 1275} \mid 1275 \\ \underline{1275} \end{array}$$

Thus $\frac{4}{17} = \frac{1}{2} + \frac{1}{23} + \frac{1}{1564}$, and also $= \frac{1}{3} - \frac{1}{25} + \frac{1}{1275}$, both of which are right.

6898. (By the EDITOR.)—Show that the average area of all the circles that can be drawn through two random points so as to lie wholly within a given circle, is two-sevenths of the area of the circle.

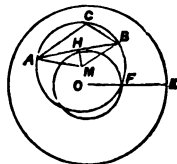
Solution by Prof. SMITH, M.A.; D. EDWARDS; and others.

Let A and B be two random points within the circle, centre O, and M the centre of a circle through A, B. The centres of all circles through A, B must lie on MH, the perpendicular at the middle point of AB, and we will suppose the centres uniformly distributed on this perpendicular.

From O as centre with radius OF, equal to the difference between OE and AM, draw a circle.

Now, while AB and AM are given in length and direction, AM being less than OE, the area of the circle OF represents the number of ways the two points can be taken, so that the circle MA will lie wholly within the given circle.

Let AB = 2x, OE = r, $\angle AMH = \theta$, and ϕ = the angle AB makes with some fixed line. Then we have AM = x cosec θ , OF = r - x cosec θ ; an element of the circle at B is $4x dx d\phi$, and an element of the perpendicular MH at M is $x \text{ cosec}^2 \theta d\theta$. The limits of θ are 0 and π ; of x, 0 and



$r \sin \theta = x'$; and of ϕ , 0 and 2π . Hence the required average is

$$\frac{\int_0^\pi \int_0^{2\pi} \pi x^2 \operatorname{cosec}^2 \theta \cdot \pi (r - x \operatorname{cosec} \theta)^2 \operatorname{cosec}^2 \theta d\theta \cdot 4x^2 dx d\phi}{\int_0^\pi \int_0^{2\pi} \pi (r - x \operatorname{cosec} \theta)^2 \operatorname{cosec}^2 \theta d\theta \cdot 4x^2 dx d\phi} = \frac{2}{3} \pi r^2 \int_0^\pi \sin \theta d\theta + \int_0^\pi \sin \theta d\theta = \frac{2}{3} \pi r^2.$$

[In the solution of the EDITOR'S QUEST. 3335, on pp. 80, 81 of Vol. XV. of the *Reprint*, it is shown that the average of such random circles that touch the given circle is *two-fifths* of the area of that circle.]

6757. (By Professor HUDSON, M.A.)—Prove that $\frac{1}{2111109}$ has no more than 18 places in its period, when converted into a circulating decimal.

Solution by Professor MOREL; G. HEFFEL, M.A.; and others.

On sait que la fraction $\frac{1}{N \cdot N' \cdot N''}$ a pour le nombre de chiffres de sa période le plus petit commun multiple des nombres de chiffres des période correspondant aux fractions $\frac{1}{N}$, $\frac{1}{N'}$, $\frac{1}{N''}$. Or, si on décompose 2111109 en facteurs premiers, on trouve $2111109 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37$.

Mais $\frac{1}{3}$ a 1 chiffre à la période, $\frac{1}{7}$ a 6, $\frac{1}{11}$ a 2, $\frac{1}{13}$ a 6, $\frac{1}{19}$ a 18, $\frac{1}{37}$ a 3, 18 étant le plus petit commun multiple des nombres 1, 2, 3, 6, et 18, a fraction donnée a bien 18 chiffres à la période.

[The number of nines that must be taken to form a number containing any given prime factor, need not be greater than the factor minus one, and, if less than this value, it must be a factor thereof. In the case of a composite factor, the number of nines must be the L. C. M. of the numbers due to its prime factors respectively. Now

$$2111109 = 3 \times 7 \times 11 \times 13 \times 37 \times 19.$$

Of these factors all but the last are factors of 999999, and 19 requires 18 repetitions of nine. Hence the circulating decimal contains 18 places.

Otherwise (by the PROPOSER).—We have $2111109 = 19 \times 111111$; and, by FERMAT'S theorem, 19 divides $10^{18} - 1$; also, 111111 divides $999999 \equiv 10^6 - 1$, and therefore divides $10^{18} - 1$; hence, since 111111 is prime to 19, 2111109 divides $10^{18} - 1$.]

6892. (By Professor MINCHIN, M.A.)—1. In the pure rolling of a circle on any plane curve, prove that the lengths of the roulettes described by

any two inverse points bear a very simple ratio to each other. [This is a particular case of a general theorem.]

2. In the pure slipping (*i.e.*, without rotation) of any curve A on any other B, prove that the radius of curvature at any point of the locus traced out by any carried point is equal to the sum of the radii of curvature of A and B at their corresponding point of contact.

Solution by E. B. ELLIOTT, M.A.; W. H. BLYTHE, B.A.; and others.

1. P, Q being the inverse points, AB the diameter on which they lie, and O the instantaneous centre of rotation, the ratio of OP to OQ is that of AP to AQ, by a property of the circle, and so is constant. Thus the elementary arcs described by P and Q in the small motion about O are in this constant ratio independent of O. Consequently the whole lengths of the P- and Q-roulettes have the same constant ratio.

2. δs being the infinitesimal arc of A from the point of contact at any instant to a proximate point whose tangent makes an angle $\delta\tau$ with that at the point of contact, and $\delta s'$ being the corresponding infinitesimal arc of B, the tracing point P moves through a distance $\delta s + \delta s'$, while its direction of motion turns through an angle $\delta\tau$. Thus the radius of curvature of its path $= \frac{\delta s + \delta s'}{\delta\tau} = \frac{\delta s}{\delta\tau} + \frac{\delta s'}{\delta\tau} = \rho + \rho'$.

6839. (By Professor CROFTON, F.R.S.)—Prove that $D^r x^r F(x) = 1.2 \dots r \left[F(x) + xF'(x) + \frac{1}{2!} D^2 x^2 F'(x) + \frac{1}{3!} D^3 x^3 F''(x) + \dots (r+1) \text{ terms} \right]$.

Solution by MARY S. MEYER; Prof. MATZ, M.A.; and others.

Supposing that

$$D^{r-1} x^{r-1} F(x) = (r-1)! \left[F(x) + xF'(x) + \frac{1}{2!} D^2 x^2 F'(x) + \dots \text{to } r \text{ terms} \right],$$

then $D^r x^r F(x) = D^r x + x^{r-1} F(x) = x D^r x^{r-1} F(x) + r D^{r-1} x^{r-1} F(x)$,

$$x D^r x^{r-1} F(x) = x [(r-1)! F'(x) + (r-2)! \cdot x F''(x) + \dots (r-m)! \cdot x^m F^m(x) \text{ to } r \text{ terms}],$$

$$D^{r-1} x^r F'(x) = (r-1)! \cdot x F'(x) + (r-2)! \cdot x^2 F''(x) + \dots + (r-m)! \cdot x^m F^m(x) \text{ to } r \text{ terms};$$

$$\therefore x D^r x^{r-1} F(x) = D^{r-1} x^r F'(x),$$

$$\therefore D^r x^r F(x) = r D^{r-1} x^r F'(x) + D^{r-1} x^r F'(x)$$

$$= r! \left[F(x) + xF'(x) + \dots + \frac{1}{(r-1)!} \cdot D^{r-2} x^{r-1} F'(x) \right] + D^{r-1} x^r F'(x)$$

$$= r! \left[F(x) + xF'(x) + \frac{1}{2!} D^2 x^2 F'(x) + \dots \text{to } (r+1) \text{ terms} \right].$$

If, therefore, the series be true for $r-1$, it is true for r ; but it is true for $r=1$, since $Dx F(x) = F(x) + xF'(x)$; hence it is true for all values of r .

6935. (By R. TUCKER, M.A.)—If $a + b + c = 0$, prove that

$$\left(\frac{b^2 - c^2}{a^2} + \frac{c^2 - a^2}{b^2} + \frac{a^2 - b^2}{c^2} \right) \left(\frac{a^2}{b^2 - c^2} + \frac{b^2}{c^2 - a^2} + \frac{c^2}{a^2 - b^2} \right) = 36 - 4(a^2 + b^2 + c^2)(a^{-2} + b^{-2} + c^{-2}).$$

Solution by J. HAMMOND, M.A.: A. McMURCHY, B.A.; and others.

$$\text{We have } \frac{b^2 - c^2}{a^2} + \frac{c^2 - a^2}{b^2} + \frac{a^2 - b^2}{c^2} \equiv - \frac{(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}{a^2 b^2 c^2};$$

$$\begin{aligned} \text{Expression} &= \frac{1}{a^2 b^2 c^2} [a^2(a^2 - b^2)(a^2 - c^2) + \text{&c.}] \\ &= \frac{a^3 + b^3 + c^3 - a^4(b^3 + c^3) - b^4(c^3 + a^3) - c^4(a^3 + b^3) + 3a^2 b^2 c^2}{a^2 b^2 c^2} \\ &= \frac{(a^3 + b^3 + c^3)^3 - 4a^4(b^3 + c^3) - 4b^4(c^3 + a^3) - 4c^4(a^3 + b^3) - 3a^2 b^2 c^2}{a^2 b^2 c^2}. \end{aligned}$$

Introducing the condition $a^3 + b^3 + c^3 = 3abc$, this becomes

$$\begin{aligned} 24 - 4a^3(b^{-3} + c^{-3}) - 4b^3(c^{-3} + a^{-3}) - 4c^3(a^{-3} + b^{-3}) \\ = 36 - 4(a^3 + b^3 + c^3)(a^{-3} + b^{-3} + c^{-3}). \end{aligned}$$

5803. (By J. J. WALKER, M.A.)—Prove that (1) the condition that the vector $la + m\beta$ shall be terminated by the circumference through the origin and the ends of the vectors a, β , is

$$2lmSa\beta + l(1-l)T^2a + m(1-m)T^2\beta = 0;$$

and (2) the condition for the circumference through the centres of the lines joining the ends of the vectors $a, \beta, la + m\beta$, containing the origin, is

$$(1 - l^2 - m^2)Sa\beta + lm(T'a + T'\beta) = 0.$$

Solution by the PROPOSER.

1. As a particular case of Question 5774 [see *Reprint*, Vol. 30, p. 94] the vector of the centre of the circumference (1) is given by

$$2\omega(Va\beta)^2 + a(Sa\beta - a^2)\beta^2 + \beta(Sa\beta - \beta^2)a^2 = 0.$$

Multiplying this by a, β successively, and taking the scalar parts, which will have $(Va\beta)^2$ as a common factor,

$$2S\omega a - a^2 = 0, \quad 2S\omega \beta - \beta^2 = 0.$$

Also, since $T(\omega - la - m\beta) = T\omega$, $2lS\omega a + 2mS\omega \beta = l^2a^2 + m^2\beta^2 + 2lmSa\beta$.

Eliminating $S\omega a, S\omega \beta$, $2lmSa\beta + l(l-1)a^2 + m(m-1)\beta^2 = 0$.

2. The vectors of the centres of the connectors of the ends of $a, \beta, la + m\beta$ are $a' = \frac{1}{2}[la + (1+m)\beta]$, $\beta' = \frac{1}{2}[(1+l)a + m\beta]$, $a'a' + m'\beta'\beta' = \frac{1}{2}(a + \beta)$,

where

$$l' = \frac{1+l-m}{1+l+m}, \quad m' = \frac{1-l+m}{1+l+m}.$$

Also $4\alpha^2 = l^2 a^2 + (1+m)^2 \beta^2 + 2l(1+m) Sa\beta,$
 $4\beta'^2 = (1+l)^2 a^2 + m^2 \beta^2 + 2(1+l)m Sa\beta,$
 $4Sa'\beta' = l(1+l)\alpha^2 + m(1+m)\beta^2 + (1+l+m+2lm)Sa\beta.$

Substituting these values in the condition

$$2lm'Sa'\beta' + l'(l'-1)\alpha'^2 + m'(m'-1)\beta'^2 = 0,$$

the result is $(1+l+m)\{(1-l^2-m^2)Sa\beta - lm(\alpha^2 + \beta^2)\} = 0,$

and $1+l+m=0$ is excluded by the supposition that the ends of $a, \beta, la+m\beta$ are not collinear.

6915. (By D. EDWARDS.)—Upon the surface of a circle and so as to lie wholly within it, two circles, each of radius equal to one-third that of the given circle, are drawn at random; prove that the chance that they will not intersect is $3\sqrt{3} + 4\pi$.

Solution by G. M. REEVES, M.A.; ELIZABETH BLACKWOOD; and others.

Let O be the centre of the given circle, P and Q the centres of the two equal circles. Draw OM parallel and equal to QP , and produce it to A . From O and M as centres, with radii each equal to OA minus the radius of the circle P , describe the arcs EHF and EKF .

Now, while PQ is given in length and direction, the area of $EHFK$ represents the number of ways the two circles can be drawn.

Let $OA = r$, radius of circle P or $Q = a$, $PQ = OM = 2x$, $\angle EMO = \theta$, area $EHFK = u$,

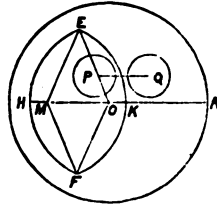
$\phi =$ the angle PQ makes with some fixed line, and $\psi = \cos^{-1}\left(\frac{a}{r-a}\right)$.

Then $OE = r-a$, $x = (r-a)\cos\theta$, $u = 2(r-a)^2(\theta - \sin\theta\cos\theta)$; an element of the circle at Q is $4x\,dx\,d\phi$. For non-intersection the limits of x are a and $r-a$, of θ are ψ and 0 , and of ϕ are 0 and 2π .

Hence, since the whole number of ways the two circles can be drawn is $\pi^2(r-a)^4$, we have

$$p = \frac{1}{\pi^2(r-a)^4} \int_a^{r-a} \int_0^{2\pi} u \cdot 4x\,dx\,d\phi = \frac{16}{\pi(r-a)^2} \int_a^{r-a} (\theta - \sin\theta\cos\theta) x\,dx \\ = \frac{16}{\pi} \int_0^\psi (\theta - \sin\theta\cos\theta) \sin\theta\cos\theta\,d\theta = \frac{2}{\pi} \left\{ \psi(1-4\cos^2\psi) + \sin\psi\cos\psi \times (1+2\cos^2\psi) \right\}.$$

If $a = \frac{1}{3}r$, $\psi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\pi$, and $p = 3\sqrt{3} + 4\pi$.

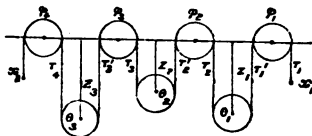


6455. (By G. F. WALKER, M.A.)—A number of uniform circular discs of radius a , but of any masses, are freely movable, in a vertical plane,

about their centres, which are fixed in a horizontal line at distances $4a$, apart. A fine rough string, of indefinite length, having two equal particles of mass m at its extremities, is laid over these circles, and uniform circular discs, each of radius a and mass $2m$, are laid on the string so as to hang between the other circles, the parts of the string not in contact with a circle being vertical. Prove that, if the system be in motion under the action of gravity, all its parts will move uniformly so long as the centres of all the discs $2m$ are below the line of fixed centres.

Solution by the PROPOSER; H. MURPHY; and others.

Let ϕ_r be the angle turned through by the r^{th} disc whose centre is fixed; x_r be the depth of the centre of r^{th} movable disc below line of centres, θ_r the angle turned through by this; T'_{r-1} , T_r the tensions of the string on either side of it; x_1 and x_2 the depths of the weights. The equations of connexion are of the form



$$\ddot{x}_1 = -a\ddot{\phi}_1, \ddot{x}_r = a\ddot{\phi}_r + a\ddot{\theta}_r = -a\ddot{\phi}_{r+1} - a\ddot{\theta}_r, \ddot{x}_2 = a\ddot{\phi}_r, \quad (2n \text{ equations}).$$

The equations of motion are of the form

$$M_r x_r \ddot{\phi}_r = a (T'_r - T_r), \quad (n \text{ equations}),$$

$$2m \cdot \frac{1}{2} a^2 \ddot{\theta}_r = a (T'_r - T_{r+1}), \quad (n-1 \text{ equations}),$$

$$2m \cdot \ddot{x}_2 = 2mg - T'_r - T_{r+1}, \quad (n-1 \text{ equations}),$$

$$m\ddot{x}_1 = mg - T_1, \quad m\ddot{x}_2 = mg - T'_n, \quad (2 \text{ equations}).$$

These are $5n$ equations to determine the $5n$ quantities [viz., the T 's and T' 's ($2n$), the $\ddot{x}$'s ($n-1$), $\ddot{\theta}$'s ($n-1$), the $\ddot{\phi}$'s (2), and the $\ddot{\phi}$'s (n)], and since they involve nothing but these and constants, they must give constant values for all the quantities. They are satisfied by all the accelerations zero, and all the T 's and T' 's equal to mg . This must therefore be the only solution, and all the parts move uniformly.

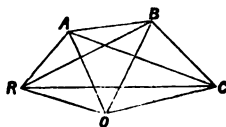
6138. (By F. MORLEY, B.A.)—If A, B, C are fixed points, and forces are denoted by OA, OB, OC; find (1) the locus of O when these forces have a resultant fixed in (α) magnitude, (β) direction, (γ) magnitude and direction; and (2) extend the problem to tetrahedra.

Solution by G. TURRIFF, M.A.; C. HARKEMA; and others.

1. Take R as origin; and let α, β, γ be vectors to A, B, C; and let $\epsilon = \alpha + \beta + \gamma$, and $RO = \delta$; then we have

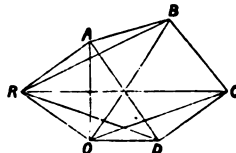
$$(OA) = \alpha - \delta, (OB) = \beta - \delta, (OC) = \gamma - \delta;$$

$$\therefore \text{resultant} = \alpha + \beta + \gamma - 3\delta = \epsilon - 3\delta.$$



- (a). $(\epsilon - 3\xi)^2 = -c^2$ (a circle).
 (β). $\epsilon - 3\xi = x\xi$, where x is variable and ξ a constant vector. The locus is therefore a line.
 (γ). The points where these two loci intersect.

2. To apply this to tetrahedra, we have simply to take another vector δ to fourth vertex, and $\omega = (RO)$ and $\epsilon = \alpha + \beta + \gamma + \delta$, and we get $(\epsilon - 4\omega)^2 = -c^2$ (a sphere), and $\epsilon - 4\omega = x\xi$ (a line), and, finally, the points where sphere and line intersect.



6117. (By Rev. W. ROBERTS, M.A.)—Find the value of $\int_0^{\pi} \frac{e E ds}{(1-e^2)(a^2-e^2)^{\frac{1}{2}}}$, where e is the excentricity of an ellipse whose major axis is 1, and E is the length of its quadrant.

Solution by Rev. T. R. TERRY, M.A.; W. H. BLYTHE, B.A.; and others.

Since
$$E = \int_0^{\frac{1}{2}\pi} (1 - e^2 \sin^2 \theta)^{\frac{1}{2}} d\theta,$$

$$\begin{aligned} \therefore I &= \int_0^{\pi} \int_0^{\frac{1}{2}\pi} \frac{(1 - e^2 \sin^2 \theta)^{\frac{1}{2}}}{(1 - e^2)(a^2 - e^2)^{\frac{1}{2}}} e ds d\theta = \frac{1}{2} \int_0^{\pi} d\theta \int_0^{\pi} \left(\frac{1 - x \sin^2 \theta}{a^2 - x} \right)^{\frac{1}{2}} \frac{dx}{1 - x} \\ &= \frac{1}{2} \int_0^{\pi} u d\theta, \text{ say; } \end{aligned}$$

therefore
$$u = \int_0^{\pi} \frac{1 - x \sin^2 \theta}{1 - x} \cdot \frac{dx}{\{(a^2 - x)(1 - x \sin^2 \theta)\}^{\frac{1}{2}}}$$

$$= \sin \theta \int_0^{\pi} \frac{dx}{\{(a^2 - x)(\operatorname{cosec}^2 \theta - x)\}^{\frac{1}{2}}} + \frac{\cos^2 \theta}{\sin \theta} \int_0^{\pi} \frac{1}{1 - x} \cdot \frac{dx}{\{(a^2 - x)(\operatorname{cosec}^2 \theta - x)\}^{\frac{1}{2}}}.$$

If in the first integral we put $a^2 - x = y^2$, and in the second $x = \frac{a^2 - z^2}{1 - z^2}$, we integrate at once, and find

$$u = \sin \theta \log \frac{1 + a \sin \theta}{1 - a \sin \theta} + \frac{2 \cos \theta}{(1 - a^2)^{\frac{1}{2}}} \sin^{-1} \frac{a \cos \theta}{(1 - a^2 \sin^2 \theta)^{\frac{1}{2}}}.$$

Hence, integrating by parts, we have

$$\int_0^{\frac{1}{2}\pi} u d\theta = \int_0^{\frac{1}{2}\pi} \left(\frac{2a \cos^2 \theta}{1 - a^2 \sin^2 \theta} + \frac{2a \sin^2 \theta}{1 - a^2 \sin^2 \theta} \right) d\theta = \frac{\pi a}{(1 - a^2)^{\frac{1}{2}}};$$

therefore
$$I = \frac{1}{2} \pi \frac{a}{(1 - a^2)^{\frac{1}{2}}}.$$

[See WILLIAMSON'S *Integral Calculus*, p. 366, Ex. 44.]

2586. (By the EDITOR.)—P is a given point outside or inside a circle whose centre is O; find the locus of another point R inside or outside the circle, such that, if P, R be joined, and lines be drawn through P, R to cut the circle in A, B and C, D, we may have

$$PR^2 = CR \cdot RD - AP \cdot PB.$$

Solution by J. HAMMOND, M.A.; BELLE EASTON; and others.

When P is outside the given circle (of radius r) and R inside, we have

$$CR \cdot RD = r^2 - OR^2, \quad AP \cdot PB = OP^2 - r^2,$$

and we have $PR^2 = 2r^2 - OP^2 - OR^2 \dots\dots\dots(1)$;

with R outside and P inside, the relation is

$$PR^2 = OP^2 + OR^2 - 2r^2 \dots\dots\dots(2).$$

In the first case, R lies on a circle whose centre is the middle point of OP; for, OP being bisected in Q, $OR^2 + PR^2 = \frac{1}{2}OP^2 + 2QR^2$,

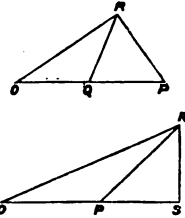
or $QR^2 = r^2 - \frac{1}{4}OP^2 = \text{constant}$

which is real when $OP < \frac{2}{3}r\sqrt{3}$.

In the second case, R lies on the polar of P; for, RS being drawn perpendicular to OP, $OR^2 - PR^2 = OS^2 - SP^2 = OP^2 + 2OP \cdot PS$;

and, substituting in (2), we have $OP \cdot OS = r^2$. In this case the locus of R is real for every position of P within the circle.

If the signs of the segments be regarded, we have $PR^2 + OR^2 = OP^2$, and R lies on a circle on OP as diameter.



6819. (By R. KNOWLES, B.A., L.C.P.)—Show that the sum to n terms of the series $1 + 5 + 13 + 29 + 61 + \dots = 4(2^n - 1) - 3n$.

Solution by ARTHUR COHEN, Q.C., M.P.; KATE GALE; and others.

$$\begin{aligned} \text{The series} &= (4-3) + (8-3) + (16-3) + \dots + (4 \cdot 2^{n-1} - 3) \\ &= 4(1 + 2 + 4 + \dots + 2^{n-1}) - 3n = 4(2^n - 1) - 3n. \end{aligned}$$

6849. (By ASUTOSH MUKHOPADHYA'Y.)—In a given triangle, the sum of the squares on the sides is equal to eight times the square on the line joining the vertex with the middle point of the base; prove that the triangle is right-angled.

Solution by E. RUTTER; G. M. REEVES, M.A.; and others.

We have $a^2 + b^2 + c^2 = 8d^2$; also $a^2 + b^2 = 2(\frac{1}{2}c)^2 + 2d^2$;
hence $a^2 + b^2 = c^2$; therefore &c.

6397. (By H. L. ORCHARD, M.A.)—A cube and a tetrahedron are inscribed in two equal spheres; show that the ratio of the sums of their edges can be expressed as a simple trigonometrical function.

Solution by E. RUTTER; W. J. C. SHARP, M.A.; and others.

Let r be the radius of the sphere; a , b the edges of the cube and the tetrahedron respectively; then

$$a = \frac{2}{\sqrt{3}} r, \quad 6a = 2\sqrt{3}r, \quad \left(\frac{b}{2\sin 60^\circ}\right)^2 + \left(\frac{r}{3}\right)^2 = r^2,$$

$$b^2 = \frac{1}{3}r^2, \quad 4b = \frac{8\sqrt{2}}{\sqrt{3}}r, \quad 6a : 4b = 6 : 4\sqrt{2} = 4\sin^2 60^\circ : \sin 45^\circ.$$

6759. (By H. G. DAWSON.)—Show that (1) the equation of an ellipse is $\rho^{-1} = (ab)^{-1} \{a^2 \sin^2 \psi + b^2 \cos^2 \psi\}$, where ρ is the radius of curvature at any point, and ψ the angle which tangent makes with axis of x ; and hence (2) that the area enclosed between quadrant of ellipse, axes, and evolute is $\frac{3(a^4 + b^4) \cdot 2a^2b^2}{32ab} \pi$; and (3) if the equation of a curve be $\rho = a \cos^n \psi$, the area between the curve, the evolute, and the normals at the points where $\psi = \frac{1}{2}\pi$, $\psi = 0$, is $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \frac{\pi a^2}{4}$.

[To the above class belongs $y = a \log \left(\sec \frac{x}{a} \right)$, its area being $\frac{1}{2}\pi a^2$].

Solution by the PROPOSER; E. RUTTER; and others.

$\rho = a = \frac{b'^2}{ab}$, where b' = semidiameter conjugate

to ρ , $\frac{ab}{b'} = OX \sin \psi$, $\frac{ab}{b'} = OY \cos \psi$,

therefore $\frac{x}{a} = \frac{b'}{b} \sin \psi$, $\frac{y}{b} = \frac{b'}{a} \cos \psi$,

therefore $(ab\rho) \left\{ \frac{\sin^2 \psi}{c^2} + \frac{\cos^2 \psi}{a^2} \right\} = 1$,

if XY be any curve; OP , OQ two consecutive normals; then O is a point on evolute, and

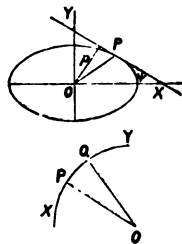
$OP = \rho$, $POQ = d\psi$, therefore $POQ = \frac{1}{2}\rho^2 d\psi$.

Hence, in above case, area between axes, evolute, and ellipse

$$= \frac{1}{2} \int_0^{\frac{1}{2}\pi} \frac{(ab)^4 d\psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^3} = \frac{3(a^4 + b^4) + 2a^2b^2}{32ab} \pi;$$

and if curve be $\rho = a \cos^n \psi$,

$$\text{Area} = \frac{1}{2} a^2 \int \cos^{2n} \psi = \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \frac{\pi a^2}{4}.$$



6364. (By W. E. WRIGHT, B.A.)—If A, B, C be the angles of a plane triangle, find the minimum value of $\cot A + \cot B + \cot C$.

Solution by P. E. BOWMAN, M.A.; W. B. GROVE, B.A.; and others.

$\omega = \cot A + \cot B + \cot C$, and $A + B + C = 180^\circ$,
 $\therefore \operatorname{cosec}^2 A \, dA + \operatorname{cosec}^2 B \, dB + \operatorname{cosec}^2 C \, dC = 0$, and $dA + dB + dC = 0$;
 hence $\operatorname{cosec}^2 A = \operatorname{cosec}^2 B = \operatorname{cosec}^2 C = \lambda$;
 consequently $A = B = C = 60^\circ$,
 and the triangle is equilateral.

The value of ω is $\sqrt{3}$, a minimum, since the maximum is infinite.

5649. (By S. TEBAT, B.A.)—From a bag containing n cards, a card is drawn but not replaced; then a card is drawn and replaced i times in succession, the sum of the numbers drawn being s ; required the probability that the x th card was first drawn.

Solution by the PROPOSER; CHRISTINE LADD; and others.

We have to find the coefficient of x^s in the expansion

$$(r + r^2 + \dots + r^n - r^n)^i = (r + r^2 + \dots + r^n)^i - i(r + r^2 + \dots + r^n)^{i-1} r^n \\ + \frac{i(i-1)}{1 \cdot 2} (r + r^2 + \dots + r^n)^{i-2} r^{2n} - \&c.$$

Let $C_{s-ix, i-t}$ be the coefficient of r^{s-ix} in the expansion of

$$(r + r^2 + \dots + r^n)^{i-t};$$

then the coefficient of r^s in the expansion of $(r + r^2 + \dots + r^n - r^n)^i$ is

$$C_{s-i, i} C_{s-x, i-1} + \frac{i(i-1)}{1 \cdot 2} C_{s-2x, i-2} - \&c. \dots\dots\dots (A).$$

The numbers $C_{s-x, i-1}$, $C_{s-2x, i-2}$, &c. are readily deduced from

$$C_{s, i} = \frac{(s-1)!}{(i-1)!(s-i)!} - \frac{i}{1} \frac{(s-n-1)!}{(i-1)!(s-i-n)!} \\ + \frac{i(i-1)}{1 \cdot 2} \frac{(s-2n-1)!}{(i-1)!(s-i-2n)!} - \&c.,$$

by making the necessary changes in s and i .

Let $N_1, N_2, \dots, N_n$ be the values of (A) when $x = 1, 2, \dots, n$; then the probability required is $\frac{N_x}{\Sigma(N)}$. Let $n = 7$, $i = 3$, $s = 12$, $x = 4$, and we find $N_1 = 25$, $N_2 = 22$, $N_3 = 22$, $N_4 = 18$, $N_5 = 22$, $N_6 = 22$, $N_7 = 25$, $\Sigma(N) = 166$, and the chance = $\frac{18}{166} = \frac{9}{83}$.

6844. (By J. L. MCKENZIE, B.A.)—Show that the equation of the envelop of a circle passing through the cusp of a cardioid, and cutting the curve at right angles, is

$$27a^2y^2(x^2+y^2)+(x^2+y^2-2ax)^3=0.$$

Solution by Prof. WOLSTENHOLME, M.A.; A. H. CURTIS, LL.D.; and others.

Inverting with respect to the pole, the inverse of the cardioid $r = a(1 + \cos \theta)$ is the parabola $r(1 + \cos \theta) = a$; the inverse of a circle through the pole cutting the cardioid at right angles is a normal to the parabola; and its envelop is

$$27(\frac{1}{2}ay^2) = 4(x - \frac{1}{2}a)^3 \text{ or } 27ar^2 \sin^2 \theta = (2r \cos \theta - a)^3,$$

$$\text{whose inverse is } 27 \frac{a^5}{r^2} \sin^2 \theta = \left(2 \frac{a}{r} \cos \theta - a\right)^3,$$

which is equivalent to the equation given in the question.

6537. (By Prof. WOLSTENHOLME, M.A.)—At any point of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, r denotes the central distance, p the perpendicular from the centre on the tangent plane, n the length of the chord intercepted by the ellipsoid on the normal, and ϕ the angle between the tangent planes at the ends of this chord; prove that

$$np = \frac{2a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2 - p^2(a^2 + b^2 + c^2 - r^2)} \dots\dots\dots(1),$$

$$2a^2b^2c^2 \tan^2 \phi = n^2p^3(2p - n) - (2a^2 - np)(2b^2 - np)(2c^2 - np) \dots\dots(2).$$

Solution by A. McMURCHY, B.A.; CHRISTINE LADD, B.A.; and others.

Let (x', y', z') be the point on the ellipsoid at which the tangent plane is drawn; and (x, y, z) the point where the normal at (x', y', z') produced cuts the ellipsoid; then we have

$$x^2 + y^2 + z^2 = r^2, \quad \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1, \quad \frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4} = \frac{1}{p^2} \dots\dots(1, 2, 3),$$

$$\frac{x - x'}{a^2} = \frac{y - y'}{b^2} = \frac{z - z'}{c^2} = np, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \dots\dots\dots(4, 5),$$

From (1), (2), (3), and (4), $x^2 = a^4 \frac{p^2(b^2 + c^2 - r^2) - b^2c^2}{p^2(a^2 - b^2)(c^2 - a^2)}$, &c. = &c.

$$x^2 = \frac{(np + a^2)^2 \{p^2(b^2 + c^2 - r^2) - b^2c^2\}}{p^2(a^2 - b^2)(c^2 - a^2)}.$$

Substitute in (5)

$$(np + a^2)^2 b^2 c^2 (b^2 - c^2) \{ p^2 (b^2 + c^2 - r^2) - b^2 c^2 \} + \dots + \dots \\ = p^2 a^2 b^2 c^2 (a^2 - b^2)(b^2 - c^2)(c^2 - a^2),$$

or $(2a^2 + np)(b^2 c^2 - b^2 c^4) \{ p^2 (b^2 + c^2 - r^2) - b^2 c^2 \} + \dots + \dots = 0$;

whence, throwing out the common factor $(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$, we obtain the stated value for np .

6828. (By Prof. TOWNSEND, F.R.S.)—A variable chord of a plane nodal cubic being supposed divided harmonically in every position, (a) by the two nodal tangents, (b) by any fixed pair of lines dividing harmonically the nodal angle; show that—

(a) It envelopes, in the former case, a conic; touching the two nodal tangents at their two points of intersection with the inflexional axis of the curve;

(b) It turns, in the latter case, round a point on the curve; the intersection, viz., of the tangents at the two points of meeting of the fixed pair of lines with the curve.

I. Solution by A. McINTOSH, B.A.; T. WOODCOCK, B.A.; and others.

In any nodal cubic $ax^3 + 3hx^2y + 3kxy^2 + by^3 = xy$, if $y = \mu x$, and if any line be drawn to intersect it in three points μ_1, μ_2, μ_3 it may be easily proved that $\mu_1\mu_2\mu_3 = -\frac{a}{b}$.

Hence, for the point μ' , from which two tangents are drawn to the curve, and for the points of contact μ , we have $\mu^2\mu' = -\frac{a}{b}$. This means that, if two lines be drawn from the origin or node to the points of contact of two tangents which intersect on the curve, these two lines form a harmonic pencil with the axes or nodal tangents.

Now, for any pair of lines which form a harmonic pencil with these two, the equation must be of the form $y^2 + 2rxy + \mu^2 x^2 = 0$, since they form a harmonic pencil with $y^2 - \mu^2 x^2 = 0$. Any line, then, which is drawn through the points of intersection of the lines $y^2 + 2rxy + \mu^2 x^2 = 0$, will have the product of two of the μ 's of its three points of intersection equal to μ^2 . But the continued product of the three μ 's is $-\frac{a}{b}$, therefore the third must be $-\frac{a}{b\mu^2}$, and this is the same as μ' .

For the first part of the question, we require to find the equation of a line which passes through the points of intersection of

$$y^2 - \mu^2 x^2 = 0 \text{ with } ax^3 + 3hx^2y + 3kxy^2 + by^3 = xy.$$

Substitute for x^2 in ax^3 the value $\mu^{-2}y^2$, and for y^2 in by^3 the value μ^2x^2 , and divide by xy .

The result, which is $a\mu^{-2}y^2 + 3hx + 3ky + b\mu^2x = 1$, is the equation of the line in question.

The envelope of this line is $4abxy = (3Ax + 3ky - 1)^2$.

This is a conic which touches the nodal tangents $x = 0$ and $y = 0$ where the line $3Ax + 3ky - 1 = 0$ meets them, but the latter line is the inflexional axis of the cubic; therefore, &c.

It may be noticed that this conic is a parabola when the nodal cubic is a circular one.

If the inflexional axis is projected to infinity, the curve reduces to $ax^3 + by^3 = cxy$, which resembles the folium of Descartes. Hence, many of the properties of nodal cubics in general may be derived from a consideration of this simple case.

II. Solution by the PROPOSER.

The equation of the conic, to the two nodal tangents as axes of coordinates, being $u_3 + xy = 0$, where $u_3 = ax^3 + by^3 + cx^2y + dy^2x$, that of the three lines from the node to its three points of intersection with any arbitrary line $lx + my = 1$ is consequently $u_3 + (lx + my)xy = 0$; which, when two of the latter divide harmonically the nodal angle $xy = 0$, must be of the form $(px + qy)(x^3 - h^2y^3) = 0$, from which it follows that then $p = a$, $h^2q = -b$, $q = (c + l)$, $h^2p = -(d + m)$, and that consequently $(c + l)(d + m) = ab$, thus showing that the envelop of the line $lx + my = 1$ is the conic $(cx + dy + 1)^2 = 4ab \cdot xy$, and therefore proving (a'); and which again, when two of the same divide harmonically the angle $x^3 - h^2y^3 = 0$, which itself divides harmonically the nodal angle $xy = 0$, must be of the form $(px + qy)[(x + hy)^2 - k^2(x - hy)^2]$, from which it follows that then $p(1 - k^2) = a$, $h^2q(1 - k^2) = b$, and consequently that the position of the third line $px + qy = 0$ is independent of the value of k , and fixed when that of h is given; hence manifestly (b'), and therefore, &c., as regards both parts of the question.

The two aforesaid general properties become manifestly transformed by reciprocation into the following:—viz., Two variable tangents to a tricuspal plane quartic being supposed to divide harmonically (a) the bitangent chord, (b) any fixed segment dividing the same harmonically; they intersect

(a') In the former case, on a conic; passing through the two extremities of the bitangent chord, along their two lines of connection with the cuspidal centre of the curve.

(b') In the latter case, on a tangent to the curve; the connector, viz., of the points of contact of the tangents to it from the two extremities of the fixed segment of the chord.

6787. (By the Rev. W. A. WHITWORTH, M.A.)—If $3n$ cyphers be placed between the digits 3 and 7, prove (1) that the number thus formed is divisible by 37; and if $3n + 1$ cyphers be placed between the digits 7 and 3, prove (2) that the number thus formed is divisible by 37.

I. Solution by G. F. WALKER, M.A.; KATE GALE; and others.

It is required to show that $3 \cdot 10^{3n+1} + 7$ and $7 \cdot 10^{3n+2} + 3$ are both

divisible by 37. The difference between any two consecutive numbers of either series is divisible by $10^3 - 1 = 999 = 37 \cdot 27$; and since the two first numbers of the two series are 37 and $703 = 37 \times 19$, we see that the theorem is true.

Since $702 = 26 \cdot 27$, we see at the same time that $2 \cdot 10^{3n+1} + 7$ and $7 \cdot 10^{3n+2} + 2$ are both divisible by 27.

II. Solution by the PROPOSER.

The following solution includes both propositions:—Write down ...000370000... with as many cyphers as you choose at each end. It is obviously divisible by 37. But if any digit be moved 3 places to the right or left, the number must be increased or decreased by a multiple of 999 which is a multiple of 37. Therefore it will remain divisible by 37. But if the 7 be moved $3n$ places to the right, there will be $3n$ cyphers between 3 and 7; and if the 7 be moved $3(n+1)$ places to the left, there will be $3n+1$ cyphers between 7 and 3. Therefore &c., both (1) and (2).

6848. (By R. KNOWLES, B.A., L.C.P.)—AB, BC are two lines perpendicular to each other, A is a fixed point, and B moves along a given straight line; prove that the envelop of the locus of C is a parabola.

Solution by A. H. CURTIS, LL.D., D.Sc.; SARAH MARKS; and others.

This question is usually solved as the converse of the property of a parabola, that the intersection of any tangent to the curve with the perpendicular let fall on it from the focus lies on the tangent at vertex. An extension of the question, obtained by substituting in it the words "inclined at a given angle" for the word "perpendicular," can be easily established as follows:—Let A be the given point, BC the given line, and AB ($=m$) perpendicular to BC, and let CD be inclined to AC at the given angle γ ; also let the polar coordinates of any point D on this line be r and θ ($=\angle BAD$); then, angle BAC being denoted by α , angle CAD $= \theta - \alpha$, and angle ADC $= \pi - (\theta - \alpha + \gamma)$, therefore

$r \sin (\theta - \alpha + \gamma) = \frac{m \sin \gamma}{\cos \alpha}$; and we seek the envelop of the line thus

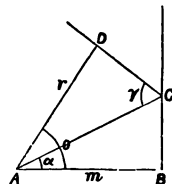
defined in polar coordinates, or of

$$2m \sin \gamma = r \sin (\theta + \gamma) (1 - \cos 2\alpha) - r \cos (\theta + \gamma) \sin 2\alpha,$$

or of $r \sin (\theta + \gamma) \cos 2\alpha + r \cos (\theta + \gamma) \sin 2\alpha = r \sin (\theta + \gamma) - 2m \sin \gamma$;

and, as this equation is of the form $L \cos \phi + M \sin \phi = R$, the envelop of which is $R^2 = L^2 + M^2$, the required result is

$$[r \sin (\theta + \gamma) - 2m \sin \gamma]^2 = r^2 [\sin^2 (\theta + \gamma) + \cos^2 (\theta + \gamma)] = r^2,$$



or $r = [r \sin(\theta + \gamma) - 2m \sin \gamma] = (x - 2m) \sin \gamma + y \cos \gamma$;
that is, a parabola whose focus is A and the directrix of which is defined
by $(x - 2m) \sin \gamma + y \cos \gamma = 0$.

When $\gamma = \frac{1}{2}\pi$, the equation of the directrix is $x - 2m = 0$; so that, in
this case, the line BC, whose equation is $x - m = 0$, is, as is geometrically
evident, the tangent to the parabola at its vertex. All that has been
proved above can without much difficulty be established geometrically.

6674. (By J. J. WALKER, M.A.)—If τ' , τ'' , τ''' are vectors parallel
to the tangents to the circumference passing through the terms of the
vectors α , β , γ , at those points respectively; show that—

$$(1) \quad \tau' = b^2(\alpha - \beta) + c^2(\gamma - \alpha), \quad \tau'' = c^2(\beta - \gamma) + a^2(\alpha - \beta), \quad \tau''' = \dots,$$

where $a = \tau(\beta - \gamma)$, $b = \tau(\gamma - \alpha)$, $c = \tau(\alpha - \beta)$;

and (2) that the vector parallel to the tangent at the term of $l\alpha + m\beta + n\gamma$
is $\tau = l\tau' + m\tau'' + n\tau'''$. (See Question 5889.)

Solution by the PROPOSER.

(1). If ω is the vector to the centre, we have

$$\tau' = V \cdot (\omega - \alpha) V(\gamma - \alpha)(\alpha - \beta) \text{ or } [A = S(\alpha - \beta)(\alpha - \gamma) \dots a = \tau(\beta - \gamma) \dots],$$

$$(Aa^2 + Bb^2 + Cc^2) \tau' = V \cdot \{Cc^2(\gamma - \alpha) - Bb^2(\alpha - \beta)\} V(\gamma - \alpha)(\alpha - \beta),$$

[see Quest. 5774, *Reprint*, Vol. XXX., p. 94]

$$= -(CA + Bb^2) c^2(\gamma - \alpha) - (AB + Cc^2) b^2(\alpha - \beta).$$

But $Aa^2 + Bb^2 + Cc^2 = 2b^2c^2 \sin^2 \theta = -2(CA + Bb^2) = -2(AB + Cc^2)$,
 θ being the angle between the connectors of the term of α with the terms
of β and γ . Hence, omitting the numerical factor, we have

$$\tau' = c^2(\gamma - \alpha) + b^2(\alpha - \beta).$$

(2). Again, $\tau = V \cdot (\omega - l\alpha - m\beta - n\gamma) V(\gamma - \alpha)(\alpha - \beta)$, with $l + m + n = 1$;

$$\begin{aligned} \text{hence } \tau &= V \cdot \{l(\omega - \alpha) + m(\omega - \beta) + n(\omega - \gamma)\} V(\gamma - \alpha)(\alpha - \beta) \\ &= V \cdot l(\omega - \alpha) V(\gamma - \alpha)(\alpha - \beta) + V \cdot m(\omega - \beta) V(\alpha - \beta)(\beta - \gamma) \\ &\quad + V \cdot n(\omega - \gamma) V(\beta - \gamma)(\gamma - \alpha) \\ &= l\tau' + m\tau'' + n\tau'''. \end{aligned}$$

5556. (By J. HAMMOND, M.A.)—Prove that

$$I \equiv \int_0^1 \int_0^1 (u-v)^{2m} (uv)^{n-1} du dv = \frac{1}{m+n} B(2m+1, n).$$

Solution by D. EDWARDES; BELLE EASTON; and others.

Put $v = uy$; then, changing the order of integration,

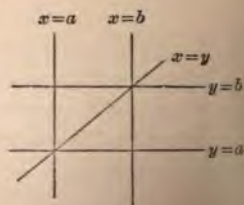
$$\begin{aligned} I &= \int_0^1 \int_0^{\frac{1}{u}} u^{2m+2n-1} (1-y)^{2m} y^{n-1} du dy, \\ I &= \int_0^1 \int_0^1 u^{2m+2n-1} (1-y)^{2m} y^{n-1} dy du + \int_1^{\infty} \int_0^{\frac{1}{u}} u^{2m+2n-1} (1-y)^{2m} y^{n-1} dy du \\ &= \frac{1}{2m+2n} \int_0^1 (1-y)^{2m} y^{n-1} dy + \frac{1}{2m+2n} \int_1^{\infty} \frac{(1-y)^{2m}}{y^{2m+n+1}} dy \\ &= \frac{1}{2m+2n} B(2m+1, n) + \frac{1}{2m+2n} \int_0^1 (1-y)^{2m} y^{n-1} dy = \frac{1}{m+n} B(2m+1, n). \end{aligned}$$

[The theorem may be extended by supposing $F(x, y)$ to be a symmetrical function of x and y , and considering its integral with respect to both x and y , the limits being independent of x and y and the same for both integrations. Or, say, the integral is

$$\int_a^b \int_a^b F(x, y) dx dy, \text{ where } F(x, y) = F(y, x);$$

$$\text{then } \int_a^b \int_a^b F(x, y) dx dy = 2 \int_a^b dx \int_a^x F(x, y) dy.$$

The proof is extremely simple, for the integrations extend in one case over the whole square, and in the other over the half square, and the symmetrical nature of $F(x, y)$ shows us that the integral is the same for each half. Considered as a particular case of this theorem, the foregoing solution would be shortened one-half.]



6961. (By T. MUIR, M.A., F.R.S.E.)—The Jacobian of the functions

$$\begin{array}{l} V_1 \equiv -ay - bz - cu - dv \\ V_2 \equiv ax - ez - fu - gv \\ V_3 \equiv bx + cy - hu - kv \end{array} \quad \begin{array}{l} V \\ V \\ V \end{array} \quad \begin{array}{l} +fy + hz - i \\ +gy + kz + l \\ \text{vanishes} \end{array}$$

being a zero-axial skew determinant condition connecting the functions.

Solution by W. J. C. SHARP

$$\begin{array}{l} \text{We have } -ay - bz - cu - dv - \\ \quad bx + cy - hu - kv \\ \quad \quad \quad dx \end{array}$$

and from these there follows

$$V_1 x + V_2 y + V_3 z + V_4 u + V_5 v = 0.$$

Eliminating x, y, z, u, v from the six equations, and extracting the square root, we have the desired relation,

$$\begin{vmatrix} a & b & c & d & V_1 \\ e & f & g & V_2 \\ h & k & V_3 \\ l & V_4 \\ V_5 \end{vmatrix} = 0.$$

[Similar functions may be easily formed of any odd number of variables, while this is impossible for an even number, a consideration which confirms the known property of zero-axial skew determinants.]

PROBABILITY AND LISTERISM. No. 1. *By the Editor.*

In the *Educational Times* for December, 1881, the following problem was proposed as Question 6929, by Dr. DONALD MACALISTER:—

“Of 10 cases treated by LISTER’s method, 7 did well, and 3 suffered from blood-poisoning; of 14 cases treated with ordinary dressings, 9 did well, and 5 had blood-poisoning: what are the odds that the success of LISTER’s method was due to chance?”

In the *Educational Times* for February, we published two solutions of this question (reproduced on pages 40–42 of this Volume),—the one by Dr. MACFARLANE, who by a brief but somewhat obscure process arrives at the conclusion that the odds are 45 to 4 that the success obtained by LISTER’s method in a given case is not due to the characteristic part of the method; the other by Dr. MACALISTER, the proposer, who finds the odds that the success obtained by LISTER’s method was not due to chance, to be nearly 3 to 2.

Since the appearance of these solutions we have received a mass of correspondence bearing thereon, from which we select for publication the following notes or remarks:—

1. Dr. MACFARLANE writes as follows:—“As the Proposer, in giving his own solution of this question, has made some observations which indicate that my solution is not fully understood, it is necessary for me to explain the view which I took of the question. I assumed that the data given were to be taken as representative of the results due to the Listerian and ordinary methods; that the relation of the former method to the latter was assumed to be known, namely, in the additional use of antiseptic precautions; and that the question was, to find the probability that a given person successfully treated by the former method would not have been successfully treated if the antiseptic precautions had been left out. The answer which I give to this question is $p - q$, and therefore $q + (p - q)$ is the odds that a person successfully treated by LISTER’s method would have been successfully treated though the antiseptic precautions had been neglected. According to the given data, interpreted as above, this is nearly 11 to 1. I fail to see how this answer

can strike an unsophisticated mind as odd. No doubt it will so strike any mind, sophisticated or unsophisticated, when it is misrepresented as meaning that the odds are more than 11 to 1 against there being anything in Listerism. My solution assumes throughout that the Listerian is superior to the ordinary method. In answer to the Proposer's objection that the solution is no answer to his question, I wish to remark that the question is not entirely free from ambiguity. 'What are the odds that the success of LISTER's method was due to chance?' does not seem a very clear way of asking, 'What are the odds that the apparent success of LISTER's method was a real success?' Dr. MACALISTER's problem, as explained by himself, is very interesting, and his solution of it, as so explained, I do not call in question. That it should have called forth a solution proceeding upon a different interpretation is, I suppose, a matter not to be regretted."

2. Mr. HUGH MACCOLL sends the following remarks:—"Dr. MACALISTER remarks that it is a good rule in Probabilities to refrain from introducing any datum of your own into the conditions of the question; and this rule he, in a somewhat sarcastic vein, charges his fellow-countryman with having broken. Now, we are all fallible, and I am by no means prepared to defend Dr. MACFARLANE's solution, which indeed I do not understand; but neither am I quite satisfied that Dr. MACALISTER has been strictly faithful to his own principles. He seems to me to have 'introduced into the conditions of the question' the datum that, *independently of the experiments*, it is an even chance (or $\frac{1}{2}$) whether LISTER's method has any advantage over the ordinary methods. This might be a perfectly legitimate assumption, if *expressly stated in the given conditions of the question*; but, as the Proposer has not so stated it, I do not think he is at liberty to take it for granted in his solution. In any case this independent or *à priori* chance is generally too important a factor to be safely left out of account in the application of the Theory of Probability to such problems as the one under discussion."

3. Miss ELIZABETH BLACKWOOD, after seeing the foregoing notes, writes thus:—"The three MACs appear to be having a very pretty 'triangular duel,' which seems, in fact, to furnish a good example of the famous fighting propensities recorded in the old song, where

'Macgregor's men, Clanronald's men, Mackenzie's men, Macfarlane's men,
With strong claymores and glittering blades, are fighting late and airy.'

I will merely remark that Dr. MACALISTER would probably feel less satisfied as to the correctness of his result, if LISTER were not the eminent man of science he is, but some superstitious old woman who, while really expert in dressing wounds, relied for protection against blood-poisoning mainly upon some mysterious charms and incantations.

"On the other hand, Dr. MACFARLANE's method of solution would lead to the perplexing paradox, that even unbroken success through numberless trials on the side of LISTER, balanced against a success of only 60 per cent. on the side of ordinary dressings, would still leave it improbable in the ratio of 3 to 2 that there was any advantage in Listerism, a conclusion which Dr. MACFARLANE is, I should say, hardly prepared to face."

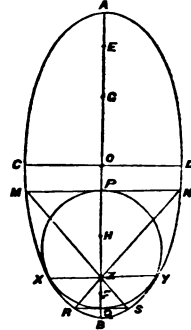
4. To these criticisms Dr. MACALISTER makes the following reply:—"The comments you have been kind enough to send me do not deal with my solution of my problem, but rather with certain matters of words, not calling for much answer. Dr. MACFARLANE agrees with me, that by his series of assumptions he has changed my problem into a different one of his own; it is hardly exact, I think, to speak of this process as 'interpre-

tation.' I unfortunately still fail to see for what reason the 'answer' he gives to his own question is ' $p-q$ ': he reasserts this,—I was anxious for explanation. The question of sophistication raises another issue: I am only justified, perhaps, in saying that, if I were told it was great odds that a patient recovering under Listerism would have equally recovered without it, I should, *knowing nothing more*, have inevitably inferred that there were great odds against there being 'anything in Listerism.' Dr. MACFARLANE admits, that in order to produce this result, he had to assume that Listerism was superior to ordinary methods; there is an appearance of discrepancy between the hypothesis and the consequence which shakes one's faith in the process of deduction. I am glad to be able to agree with him that my question is not 'a very clear way of asking, "What are the odds that the apparent success of LISTER's method was a real success?"' But, then, I did not mean to ask this. If a patient does well, I should call that a real success. I don't know what an apparent success means in this connexion. After consideration, I don't think I can put my question in clearer or more exact terms. LISTER's method 'succeeded': this observed event might, on the one hand, be due to something in Listerism, which 'commands success'; on the other hand, it might be due to accident, other than a special virtue in Listerism, or, as we all say, to chance. What are the odds that it was due to chance? I purposely translated my question into one of balls and urns, in order that we might have 'dry light' upon the subject; perhaps the comments would have been more edifying if they had dealt with the problem in that form. 'Not even the youngest is infallible,' is a maxim which Mr. MACCOLL would doubtless join me in approving; but, though the general proposition of my fallibility is entirely true, the particular illustration he gives is not, I think, very apt. On looking over my own solution, I do not find that I have tacitly or otherwise made an unwarranted assumption of the kind he mentions. In comparing successes and failures under Listerism to white or black balls in an urn respectively, *nothing in favour of or against Listerism being known*, I made no assumption other than that there were only *two possible events*, viz., success or failure, in any given case. To say then that, *a priori* (i.e. introducing no after-knowledge into the question), the chance of success or of failure is $\frac{1}{2}$, is surely not merely a just assertion, but the only one possible. Is it not the fundamental principle of the mathematical theory of probabilities that, 'if on a given trial, there is no reason to expect one event rather than another, the events are said to be equally possible'? And is it not just because nothing about the *a priori* advantage of Listerism is stated among the conditions of the question, that I must say, *a priori*, the chance of success or of failure is $\frac{1}{2}$? 'This independent or *a priori* chance' is, indeed, a very important factor *when you know it*. When you don't know it, does Mr. MACCOLL mean to say that the problem cannot have any solution? If it can have a solution, what would he take for the *a priori* chance? Mathematical probability measures our knowledge (or ignorance) of events so far as it goes; it has little to do with the events in themselves, or as known to an omniscient observer. (See LESLIE ELLIS's *Papers*, pp. 1—11.) Miss ELIZABETH BLACKWOOD has perhaps not read my solution: there is no symbol in it representing Mr. LISTER's science. For algebraical purposes I might substitute MUMBO JUMBO for LISTER throughout, as I substituted the letter A, and no step of the reasoning on which alone the result depends would be altered. I fear Miss BLACKWOOD has not verified her very amusing quotation."

6970. (By Dr. CURTIS.)—An ellipse is placed with its minor axis horizontal; find (1) the lines of quickest, or slowest, descent from any point on the major axis to the curve, or *vice versa*; (2) the condition that the quickest, or slowest, path should be along the major axis.

Solution by Prof. TOWNSEND, F.R.S.; Prof. GENÈSE, M.A.; and others.

If AB and CD be the axes, major and minor, of the ellipse, O its centre of figure, E and F its real foci, AG and BH the axial diameters of its circles of curvature at A and B, and PQ the axial diameter of any circle PXQY, having its centre on AB and having double contact with the curve at two points X and Y connecting by a horizontal chord XY parallel to CD. Since then, for the upper P of the two points P and Q, the two non-axial vectors PX and PY are manifestly, when X and Y are real, those of quickest descent from P to the curve, and that whatever be the inclination of AB to the horizon; while, on the contrary, when X and Y are imaginary, the axial vector PB is that possessing the property: hence, for every position of P above the point H on the transverse axis AB of the curve, the two vectors of quickest descent PX and PY are non-axial, while, for the point H itself, as well as for every position of P below it on AB, they coincide with the axial vector PB.



For any given position of P above H on AB, the chord of double contact XY, as well as the axial diameter PQ, of the circle PXQY, may be determined readily as follows. Through P draw the horizontal chord MN parallel to the minor axis CD of the curve, and then from P measure downwards on the major axis AB the distance PZ such that PZ : PM or PN = OA : OE or OB : OF; the point Z thus determined is the middle point of the required chord of contact XY, and its connectors with M and N intersect the curve again at two points R and S which connect horizontally through the lower extremity Q of the required diameter PQ. The proof of this is evident from the familiar geometry of the ellipse.

6928. (By the EDITOR.)—Show that the values of x in the equations

$$\frac{1+xy}{a} = \frac{x+y^2}{b} = \frac{x^2+y}{c} = \frac{1-x^3}{a-cx} \dots\dots\dots(1),$$

are given by $(a^2-bc)x^4 + (c^2+ab)x^3 - 4acx^2 + (a^2+bc)x + (c^2-ab) = 0 \dots\dots(2)$;

and that, when $a = 16$, $b = 17$, $c = 18$, equation (2) and its roots are

$$25x^4 - 298x^3 + 576x^2 - 281x - 26 = 0 \dots\dots\dots(3),$$

$$-0.079154299618, 1.060498774748, 1.283137095508, 9.655518429362.$$

6972. (By H. STEWART, M.C.P.)—Solve the equations $x^2 + yz = a$, $y^2 + zx = b$, $z^2 + xy = c$ [which by putting $y = ux$, $z = vx$, may be reduced to the system given in the foregoing Question 6928.]

I. Solution by J. HAMMOND, M.A. ; Prof. MATZ, M.A. ; and others.

1. It is clear that the elimination of either x or y will give a quartic equation to find the other, also that these quartics are convertible by interchanging b and c .

From (1), $y = \frac{ax^2 - c}{cx - a}$, whence $b(x^2 - 1)(cx - a) = x(ax - a)^2 + (ax^2 - c)^2$,

which gives (2). If we call a root of (2) $F(a, c, b)$, we have

$$x = F(a, b, c), \quad y = F(a, c, b),$$

and the form of (2) shows that $x^{-1} = F(c, b, a)$, $y^{-1} = F(b, c, a)$.

Let $y' = F(c, a, b)$ then (1) gives, on interchanging a and c ,

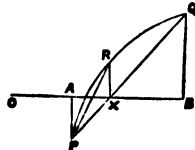
$$y' = (cx^2 - a) + (ax^{-1} - c) \text{ or } xy' = y,$$

so that $\frac{y}{x} = F(c, a, b)$, and $\frac{x}{y} = F(b, a, c)$.

With the given values the quartic becomes (3), say $f(x) = 0$; and because of the occurrence of two nearly equal roots in (3) it will be easier to find the roots of $f'(x) = 0$, which will be found to lie in the order 0, R, 1, 1.1, R, 1.2, 7, R, 8; and afterwards, by a few easy trials, the roots of (3) will be seen to follow the order -1, R, 0, 1, R, 1.1, 1.2, R, 2, 9, R, 10.

2. The ordinary or Newtonian method of approximation consists, as is well known, in taking the tangent to the curve $f(x) = y$ instead of its arc, and the distance of the foot of the tangent from the origin for a root of the equation $f(x) = 0$. We may with equal propriety take the chord for the arc. Suppose then that $a = OA$, $b = OB$ are two values of x such that only one root of the equation $f(x) = 0$ lies between them, and let $f(a) = -\alpha = \Delta P$, $f(b) = \beta = QB$; it is evident from Geometry that if $OX = x$,

$$\frac{x-a}{a} = \frac{b-x}{\beta} = \frac{b-a}{a+\beta}.$$



Thus, if a and b be approximate solutions of $f(x) = 0$, $a + \frac{(b-a)a}{a+\beta}$

is a closer approximation. A second approximation is found by taking the chord RP. In this way we are always, even in the case where a and b are not very near the required root, led to a value of x between a and b , and therefore nearer than one of them, generally nearer than both. In the figure, since OB is too great to be considered an approximation, OX is not nearer the root than OA , but the intersection of RP is; so that ultimately we arrive at a very close approximation.

The connection between this method and Newton's is readily shown.

Calling $b, a + \Delta a$, the approximation is [since $a = -f(a)$] $a - \frac{\Delta a \cdot f(a)}{\Delta f(a)}$,

and Newton's method consists in making Δa a differential instead of a finite difference. The present method has long been used, under the name of "Proportional Parts," for the solution of such equations as $\log x = a$.

3. To apply the rule to the approximate solution of (3): first, to find the negative root, change the sign of x , and our equation is

$$25x^4 + 298x^3 + 576x^2 + 281x - 26 = 0, \text{ say } \phi(x) = 0,$$

which has a root between 0 and 1, and the first Newtonian approximation to it is $\frac{35}{47} = .09...$, which is too great; it will be found that .08 is also too large and that .07 is too small, the values being

$$\phi(.08) = .32, \phi(.07) = -.3404..., .07 + \frac{.01 \times 3404}{3724} = .0791...;$$

$$\phi(.0791) = -.02051..., .0791 + \frac{.0009 \times 2051}{34051} = .0791542...,$$

$$\phi(.0791542) = -.0000376..., .0791542 + \frac{.0008458 \times 376}{3200376} = .079154299...;$$

$$\phi(.079154299) = -.000000233..., .0791542 + \frac{.000845701 \times 233}{320000233} = .079154299612...,$$

$\phi(.079154299613) = -.00000000185...$, and this corrected as before gives .0000000000050 for the correction and .079154299618 for the root.

This is a trifle greater than the true root, the value of

$$\phi(.079154299618) \text{ being } + \frac{23}{10''} \text{ very nearly.}$$

It may be noticed that the value of $\phi(x)$ need only be roughly estimated; if accurate values of $\phi(x)$ had been used instead of the approximate values given above, the approximation to the root would not have been more rapid by a single figure, though the labour would have been greatly increased.

The value .08 has been used throughout as a higher limit, but the approximation would have been rather more rapid if some other value such as .0792 had been used. Notice also that we only want to calculate $\phi(x)$ at each step, whereas Newton's method also requires the calculation of $\phi'(x)$.

Next, to find the two roots between 1 and 2, diminishing the roots of (3) by unity, we have

$25x^4 - 198x^3 - 168x^2 + 77x - 4 = 0$, say $\psi(x) = 0$ (4),
one root of which lies between 0 and .1 and another between .2 and 1.

$$\text{Here } \psi(0) = -4, \psi(.1) = 1.8..., 0 + \frac{.1 \times 4}{5.8} = .068...;$$

$$\psi(.06) = -.027244, \psi(.07) = .4994..., .06 + \frac{.01 \times 272}{5266} = .0605...;$$

$$\psi(.0605) = .0000668010015625, .06 + \frac{.0005 \times 272440}{273108} = .060498777...;$$

$$\psi(.06049877) = -.0000002588..., \\ .06049877 + \frac{.00000123 \times 2588}{670598015} = .060498774746...$$

$$\psi(.060498774746) = -.000,000,000,09795..., \\ .060498774746 + \frac{.000001225254 \times 9795}{6680109951} = .06049877474794...$$

The first Newtonian approximation to this root is $\frac{4}{77} = .05...$, and is not so close as the .068 of the present method. But if we neglect cubes of x in $\psi(x)$ and solve the equation $168x^2 - 77x + 4 = 0$, we have

$$x = \frac{77 \pm (77^2 - 16 \times 168)^{\frac{1}{2}}}{2 \times 168} = \frac{77 \pm 57}{2 \times 168}$$

to the nearest integer; and $\frac{67}{168} = .3...$, $\frac{1}{168} = .06$ nearly.

Taking $\cdot 3$ and $\cdot 2$ for limits to the third root, we have

$$\psi(\cdot 3) = -3.8635, \psi(\cdot 2) = 3.136, \cdot 2 + \frac{\cdot 1 \times 31360}{69995} = \cdot 24\dots,$$

and this, though nearer the true root than $\cdot 2$, is not as near as $\cdot 3$.

If we take $\cdot 28$ as the lower limit,

$$\psi(\cdot 28) = \cdot 195968, \cdot 28 + \frac{\cdot 02 \times 195968}{4059468} = \cdot 28098\dots$$

But the approximation to this root proceeds very slowly, whatever limits we start from; even starting with

$$\psi(\cdot 283137) = \cdot 0000060631\dots, \psi(\cdot 2831371) = -\cdot 0000003418\dots,$$

$$\cdot 283137 + \frac{\cdot 0000001 \times 60631}{64049} = \cdot 283137094\dots,$$

and we only get two additional figures.

To compensate for this the approximation to the greatest root of (3) is very rapid. In fact, after arriving at

$$f(9.655518) = -\cdot 007519350, f(9.655519) = \cdot 009993517\dots,$$

we get for the next approximation $9.655518429361\dots$, which only differs from the true root by a very small fraction.

It would be interesting to investigate the rapidity of approach, but such an investigation would unduly lengthen the present solution. In finding any particular root of any particular equation, a few trials will generally suffice to show whether the rule is unsuitable, because of the slowness of convergence, or not.

II. Solution by the PROPOSER.

This mode of solution of Dr. PELL's famous set of equations—proposed by him, in the 17th century, as is well-known, through Col. TIRUS, as a sort of challenge-problem, to Dr. WALLIS, and often discussed since WALLIS's time—is perhaps as simple as the problem admits of. We add a solution of the numerical form, as exemplifications of convenient *working* methods of the complete and elegant processes of STURM and HORNER. The contractions here adopted were first set forth by the late J. R. YOUNG, first in his *Mathematical Dissertations*, and subsequently in the second edition of his *Theory of Equations*; and, by aid hereof, STURM's theorem may be readily applied to equations of any reasonable degree.

The formation of STURM's functions by this method is as follows:—

$$f(x) = 25x^4 - 298x^3 + 576x^2 - 281x - 26, f_1(x) = 100x^3 - 894x^2 + 1152x - 281;$$

$$\begin{array}{r} 100 \quad | \quad -894 \quad + \quad 1152 - 281 \\ -298 \quad | \quad -1152 \quad + \quad 843 + 104 \\ \hline 266412 - 343296 + 83738 \\ -115200 + 84300 + 10400 \end{array}$$

$$\text{Dividing by 2) } 151212 - 258996 + 94138$$

$$\begin{array}{r} f_2(x) \dots\dots 75606 - 129498 + 47069 \\ \hline 75606 + 129498 - 47069 \\ 100 - \quad 894 + \quad 1152 - 281 \end{array}$$

$$\begin{array}{r} -4553497 \quad | \quad -589668754506 + 214328550293 \\ 630050 \quad | \quad -29655823450 \\ 476355603 \quad | \quad 548761654656 - 133855924443 \end{array}$$

$$\text{Dividing by 50) } 70562923300 - 80472625850$$

$$f_3(x) \dots\dots 1411258466 - 1609452517$$

STURM's functions are, accordingly, $f(x)$ and $f_1(x)$ as hereinbefore given,

$$f_2(x) = 75606x^2 - 129498x + 47069,$$

$$f_3(x) = 1411258466x - 1609452517, \quad f_4(x) = +.$$

Seeing that the signs of the leading terms of these functions present no variations, we infer that the roots are all real; and, as the signs of the final terms present three variations, it follows that three of the roots are positive and consequently one negative. By BRÉ'S limit (YOUNG'S *Theory of Equations*, Arts. 89, 90) it is seen that the roots are contained within the limits (13, -2); and their situations are given by the following series of signs, which STURM's functions $f(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$ present, respectively, corresponding to the values -1, 0, 1, 2, 9, 10 for x :-

+ - + - +, - - + - +, - + - - +, - - + + +,
- + + + +, + + + + +.

The several numbers of variations being, in order, 4, 3, 3, 1, 1, 0, the places of the roots are thus seen to be (0, -1), (1, 2), (1, 2), (9, 10), the separation of the two roots in the interval (1, 2) being easily ascertained by HORNER'S process of evolving the successive decimals. We add here, as specimens of a convenient form of arranging HORNER'S process, the developments of two of these roots, of which the second (B) is the development, from equation (3), of the root in the interval (9, 10), and the first, (A), is the development, from equation (4), of the *decimal* part of the second root in the interval (1, 2), wherein Mr. HAMMOND'S method of approximation "proceeds very slowly."

(A)	- 6.72	3.08	-16 (1.283137095508...
-7.72 ...	- 1.544	-1.6528	.28544
	- 8.264	1.4272	.12544
	- 1.504	- .3008	- .11760128
	- 1.464	- .5264	783872
	-11.232	- .943616	-7490833119
-7.04 ...	- .5632	-1.470016	347886881
	-11.7952	- .44544	-263707032
	- .5568	-2.458176	94179849
	- .5504	- .38768373	-76162673
	-12.9024	-2.496944373	18017176
-6.797 ...	-20391	61146	-17774649
	-12.922791	-2.535773892	242527
	-20382	-1296423	-228539
	-20373	-2.537070315	13988
	-12.963546	- .68	-12697
-6.788 ...	679	-2.53836681	1293
	-12.974225	- .38897	-1270
	-1358	-2.53875578	21
	-12.9656...	-2.5391448	-20
	-2	- .908	-12, 13.
	-12.9658	-2.5392356	
	-4	-2.539326	
-12.97...			

(B) 23·04	-11·24	-1·04 (9·65551842936153
-2·92 -26·28	-29·16	-363·60
-3·24	-40·40	-364·64
54·72 } 1	492·48	326·47488
135·72	422·92	-38·16512
187·20	121·2048	34·30659625
24·68... 14·808	544·1248 } 2	-3·85852375
202·008	9·1008	3·495420110625
15·168 } 2	674·4304	-363103639375
15·528	11·701525	350193681925
232·704	686·131925 } 3	-12909957450
26·53.....1·3265	66450	7005082942
234·0305	697·899900	-5904874508
1·3290 } 3	1·184122125	5604100502
1·3315	699·084022125 } 4	-300774006
236·6910	667250	280205822
26·685... 133425	700·26881150	-20568184
236·824425	11855235	14010293
133450 } 4	700·38736385	-6557891
133475	668	6304632
237·09135	700·50592289	-253259
26·7..... 1335	237132	210154
237·1047 } 5	700·50829421	-43105
1335	700·51066553	42031
237·1314	189706	-1074
3 } 6	700·51256269	701
237·1317	700·51445975	-373
237·132	9485	350
	700·5145546 } 8	-23
	700·51465	21
		= 15

6937. (By J. HAMMOND, M.A.)—Find in how many different ways a different things can be given to b different people.

Solution by the Rev. W. A. WHITWORTH, M.A.; J. O'REGAN; and others.

Each thing can be disposed of in b ways, and therefore all in b^a ways (*Choice and Chance*, Prop. ii. Cor.). If, however, it is implied that no person is to be left empty, the result will be $a!$ times the coefficient of x^a in the expansion of $(e^x - 1)^b$ (*Choice and Chance*, Prop. xxii., p. 84).

6900. (By ELIZABETH BLACKWOOD.)—If A, B, C, D are four candidates whose respective chances of passing a certain examination are a, b, c, d ; show that the chance that at least two out of the four will pass is $3abcd - 2\sum abc + \sum ab$, where $\sum abc$ denotes combinations of a, b, c, d , three at a time, which, when $a = b = c = d$, becomes $6a^2 - 8a^3 + 3a^4$.

Solution by W. H. BLYTHE, B.A.; G. HEPPEL, M.A.; and others.

The chance they all pass is $abcd$; that A, B, C pass, and D fails, is $abc(1-d)$, and there are four terms similar to this; the chance that A and B pass, and C and D fail, is $ab(1-c)(1-d)$, and there are six terms similar to this. If either of these events happen, two at least pass; hence, adding, we obtain the given result.

If $a' \equiv 1-a$, the chance of two at least passing may be otherwise written

$$1 - a'b'c'd' \left(1 + \sum \frac{a}{a'}\right).$$

6930. (By J. GRIFFITHS, M.A.)—Show (1) that, if

$$\frac{a}{a'} \sin \theta \sin \theta' + \frac{b}{b'} \cos \theta \cos \theta' = 1, \text{ where } a^2 - b^2 = a'^2 - b'^2,$$

then
$$\int (a'^2 \cos^2 \theta + b'^2 \sin^2 \theta)^{\frac{1}{2}} d\theta - \int (a'^2 \cos^2 \theta' + b'^2 \sin^2 \theta')^{\frac{1}{2}} d\theta'$$

$$= [(a \sin \theta - a' \sin \theta')^2 + (b \cos \theta - b' \cos \theta')^2]^{\frac{1}{2}} + \text{constant};$$

also (2) give a geometrical interpretation of the result; and (3) deduce therefrom GRAVES' theorem (SALMON'S *Conics*, 6th ed., p. 377).

I. Solution by J. J. WALKER, M.A.

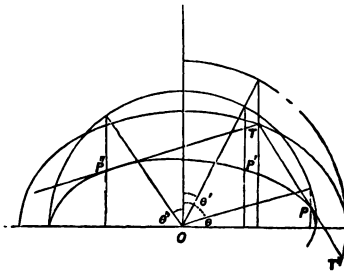
It is assumed that $a > a'$; then $ab' < a'b$, so that angles Φ, ϕ may be found, such that

$$\cos \Phi = \frac{ab'}{a'b}, \quad \cos \phi = \frac{b'}{b};$$

and the relation (1) giving

$$\sin \theta \sin \theta' \cos \Phi + \cos \theta \cos \theta' = \cos \phi$$

shows that Φ is the angle opposite ϕ in the spherical triangle having as sides θ, θ', ϕ . Suppose Θ, Θ' the other two angles, opposite θ, θ' , which are to vary so that Φ and ϕ remain constant, giving $\cos \Theta d\theta + \cos \Theta' d\theta = 0$.



$$\begin{aligned} \text{Now} \quad & \{ (a \sin \theta - a' \sin \theta')^2 + (b \cos \theta - b' \cos \theta')^2 \}^{\frac{1}{2}} \\ &= a' \left\{ \left(\frac{\cos \Phi}{\cos \phi} \sin \theta - \sin \theta' \right)^2 + \left(\frac{\cos \theta}{\cos \phi} - \cos \theta' \right)^2 (1 - e'^2) \right\}^{\frac{1}{2}}, \\ & \quad \left[\text{where } e'^2 = \frac{a'^2 - b'^2}{a'^2} = \frac{\sin^2 \Phi}{\sin^2 \phi} \right], \end{aligned}$$

that is $= a' \tan \phi \cos \theta \cos \theta'$ or $a' e'^2 \sin \theta \sin \theta' \sin \phi - a' \tan \phi \cos \Phi$, the latter term being constant. Hence, immediately, by the addition theorem of elliptic integrals of the second kind, the result (1) follows.

2. The tangent at the point on the ellipse (a', b') which has θ as its angular parameter is $\frac{x}{a'} \sin \theta + \frac{y}{b'} \cos \theta = 1$, so that, if θ' is the angular parameter of a point in which it meets the coaxial ellipse (ab) ,

$$\frac{a}{a'} \sin \theta \sin \theta' + \frac{b}{b'} \cos \theta \cos \theta' = 1.$$

Also $\{ (a \sin \theta \dots)^2 + \dots \}^{\frac{1}{2}}$ is obviously the length of the intercept on the tangent between these points; and $a'E(\theta) - a'E(\theta')$ will be the arc on the ellipse $(a'b')$ between the points whose angular parameters are θ, θ' . Thus the interesting extension of GRAVES's theorem is obtained: the difference between the length of an arc of an ellipse (P, P') defined as above, and the intercept on the tangent (PT) , also as so defined, is constant, if the two ellipses are confocal.

3. GRAVES's theorem obviously follows by considering two points (θ, θ'') , (P, P'') on the inner, having a common corresponding point θ' (T) on the outer confocal ellipse; attention being paid to the signs of the intercepts on the tangents, and of the angular parameters.

II. Solution by the PROPOSER.

1. Taking the expression

$$t^2 = (a \sin \theta - a' \sin \theta')^2 + (b \cos \theta - b' \cos \theta')^2,$$

where
$$\frac{a}{a'} \sin \theta \sin \theta' + \frac{b}{b'} \cos \theta \cos \theta' = 1,$$

we have
$$dt = \frac{aa' \cos \theta \cos \theta' + bb' \sin \theta \sin \theta'}{(a'^2 \cos^2 \theta' + b'^2 \sin^2 \theta')^{\frac{1}{2}}} d\theta - (a'^2 \cos^2 \theta' + b'^2 \sin^2 \theta')^{\frac{1}{2}} d\theta'.$$

Hence (since $a'^2 - b'^2 = a^2 - b^2$) the theorem depends upon the identity $(aa' \cos \theta \cos \theta' + bb' \sin \theta \sin \theta')^2 = (a'^2 \cos^2 \theta + b'^2 \sin^2 \theta)(a'^2 \cos^2 \theta' + b'^2 \sin^2 \theta')$.

2. As geometrical results, both GRAVES's and MACCULLAGH's theorems follow.

3. From $t^2 = (a \sin \theta - a' \sin \theta')^2 + (b \cos \theta - b' \cos \theta')^2$, where

$$\frac{a}{a'} \sin \theta \sin \theta' + \frac{b}{b'} \cos \theta \cos \theta' = 1, \quad \text{and} \quad a'^2 - b'^2 = a^2 - b^2,$$

we obtain

$$\left. \begin{aligned} \frac{a^2}{c^2} - \frac{ab}{c'^2} t &= \frac{a}{c'} \sin \theta \sin \theta', & -\frac{b^2}{c^2} + \frac{ab}{c'^2} t &= \frac{b}{c'} \cos \theta \cos \theta', \\ 2c \sin \theta &= L^{\frac{1}{2}} + M^{\frac{1}{2}}, & 2c \sin \theta' &= L^{\frac{1}{2}} - M^{\frac{1}{2}}, \end{aligned} \right\} \dots\dots\dots (A),$$

where $a^2 - a'^2 = c'^2 = b^2 - b'^2$, $a^2 - b^2 = c^2 = a'^2 - b'^2$,

$$L = c^2 - \frac{2b}{c'}(a + a')t + (a + a')^2, \quad M = c^2 - \frac{2b}{c'}(a - a')t + (a - a')^2;$$

also, from (A), we have

$$\begin{aligned} \frac{d\theta}{(a'^2 \cos^2 \theta + b'^2 \sin^2 \theta)^{\frac{1}{2}}} &= \frac{d\theta'}{(a'^2 \cos^2 \theta' + b'^2 \sin^2 \theta')^{\frac{1}{2}}} \\ &= \frac{dt}{c^2 (\sin^2 \theta' - \sin^2 \theta)^{\frac{1}{2}}} = \frac{dt}{-(LM)^{\frac{1}{2}}} \dots\dots\dots (B). \end{aligned}$$

From (B) we obtain another proof of the addition theorem for the second kind of elliptic integrals; thus, writing

$$\begin{aligned} \int_0^u \frac{d\theta}{(1 - c^2 \sin^2 \theta)^{\frac{1}{2}}} &= u, & \int_0^v \frac{d\theta'}{(1 - c^2 \sin^2 \theta')^{\frac{1}{2}}} &= v, & \int_0^u (1 - c^2 \sin^2 \theta)^{\frac{1}{2}} d\theta &= E(u), \\ \int_0^v (1 - c^2 \sin^2 \theta')^{\frac{1}{2}} d\theta' &= E(v); \end{aligned}$$

and, putting $a' = 1$, we see that

$$c^2 (\sin^2 \theta' - \sin^2 \theta) du = dt$$

is equivalent to

$$(\Delta^2 am u - \Delta^2 am v) du = dt,$$

or

$$\Delta^2 am u du - \Delta^2 am v dv = dt, \text{ since } du = dv;$$

that is,

$$E(u) - E(v) = t + \text{constant}.$$

4. The formula $2c \sin \theta = L^{\frac{1}{2}} \pm M^{\frac{1}{2}}$ gives

$$t = \frac{ab'c' (a'^2 \cos^2 \theta + b'^2 \sin^2 \theta) \pm c^2 c' \sin \theta \cos \theta (a'^2 \cos^2 \theta + b'^2 \sin^2 \theta)^{\frac{1}{2}}}{a'^2 b^2 - c^2 c'^2 \sin^2 \theta},$$

or

$$t = \tan am(u \mp v) \cdot \Delta am u \cdot \Delta am v,$$

on reduction to the standard forms by putting

$$a' = 1, \quad \frac{b'}{b} = \cos am(u \mp v), \quad \frac{c'}{b} = \sin am(u \mp v),$$

$$b = (1 - c^2)^{\frac{1}{2}} \sec am(u \mp v), \text{ and } a = \Delta am(u \mp v) \sec am(u \mp v).$$

[Mr. GRIFFITHS states, that he has recently been informed by Mr. S. ROBERTS that a deduction of GRAVES's theorem analytically is to be found in Note viii. of QUIMPER's "*Elliptische Functionen*." In this note QUIMPER makes use of the addition-theorem for elliptic integrals of the second kind, and so proceeds in a different manner from that sketched out above.]

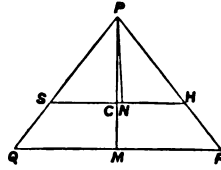
6753. (By R. TUCKER, M.A.)—PSQ, PHR are two focal chords of an ellipse inclined at an angle ω ; if the centre of the ellipse be the centroid of the triangle PQR, prove that the area of the triangle is $\frac{1}{2}b^2 \tan \frac{1}{2}\omega$.

Solution by J. O'REGAN; R. KNOWLES, B.A., L.C.P.; and others.

Since PM bisects both SH and QR, these lines must be parallel; and $PM = \frac{1}{2}PC$; hence $QH = \frac{1}{2}SH = 3ae$; also the perpendicular from P on QR = $\frac{1}{2}PN$; therefore, putting $CN = x$, $PN = y$, the area $PQR = \frac{1}{2}ae \times PN = \frac{1}{2}aey$; but, from the triangle SPH, whereof the sides are $a+ex$, $a-ex$, $2ae$, we find

$$\tan \frac{\omega}{2} = \frac{e(a^2 - x^2)}{b} = \frac{aey}{b^2};$$

therefore Area $PQR = \frac{1}{2}b^2 \tan \frac{1}{2}\omega$.



6987. (By B. REYNOLDS, M.A.)—Show that (1) 8 queens can be placed on an ordinary chess-board, so that no two can take each other, in 12 essentially different classes, forming, with permutations, 92 different sets, of ways: (2) 7 queens can be placed on a 7-square board in 6 different classes, and 40 different sets, of ways, so as not to be able to take each other.

Solution by G. HEFFEL, M.A.; ELIZABETH BLACKWOOD; and others.

1. Let the q^{th} square of the p^{th} row be denoted by the symbol $\frac{p}{q}$. Then a queen on $\frac{p}{q}$ will take a queen on $\frac{r}{s}$, by a castle's move, if either $p = r$ or $q = s$; and by a bishop's move, if either $p+q = r+s$ or $p-q = r-s$. The first part of the question is therefore reduced to finding eight fractions with numerator and denominator not greater than eight, such that no two numerators, no two denominators, and no two results, either of numerator plus denominator or numerator minus denominator, shall be identical. It will be found that there are altogether 92 such sets, which range themselves in the following 12 essentially different classes, whereof all but J may be permuted 8 times, and J may be permuted 4 times, thus giving a total of 92 ways.

A $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$	G $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$
B $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$	H $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$
C $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$	I $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$
D $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$	J $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$
E $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$	K $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$
F $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$	L $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7} \frac{7}{8}$

2. In the case of 7 queens on a 7-square board, the six essentially different ways are as follows:—

A $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7}$	D $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7}$
B $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7}$	E $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7}$
C $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7}$	F $\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{4}{5} \frac{5}{6} \frac{6}{7}$

Of these A, B, C, F allow of 8 permutations each, and D, E allow of 4 each. Hence the total number of ways is 40.

II. Solution by J. J. WALKER, M.A.

This very interesting addition to the known properties of the ovals may be proved as follows by means of the usual biradial equation, $mr + l' = nc$ (A, B), where l', m^2, n^2 may be taken as equal to OA, OB, OC, or α, β, γ .

Starting from the known property $AG : BG = lr : mr'$,

whence

$$AG = lrc + (lr + mr'),$$

$$BG = mr'c + (lr + mr') rr' = (lr + mr')^2 AG \cdot BG + lmc^2;$$

by a property of any plane triangle,

$$(lr + mr') N^2 = mr'^2 + lrr'^2 - (lr + mr') AG \cdot BG = (mr + lr') rr' - \dots,$$

or

$$N^2 = \{n(lr + mr') + lmc - 1\} AG \cdot BG.$$

$$\begin{aligned} \text{Now } OG &= OA + AG = l^2 + lrc + (lr + mr') = l^2 + lr(m^2 - l') + (lr + mr') \\ &= lm(mr + lr') + (lr + mr') = lmc + (lr + mr'); \end{aligned}$$

$$\text{and } 1 + CG + OG = OC + OG = n^2 + OG = n(lr + mr') + lmc,$$

whence

$$CG + OG = n(lr + mr') + lmc - 1;$$

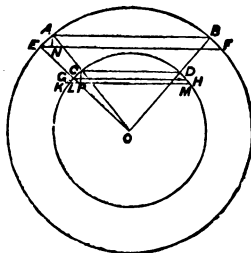
and, substituting in the value for N^2 above, the formula in the Question follows.

It may be observed that it has been incidentally shown, in the investigation which precedes, that $OG^2 : AG^2 = OB \cdot OC : r^2$, from which and the analogous relations, $OG^2 : AG \cdot BG \cdot CG = OA \cdot OB \cdot OC : rr'r''$.

6910. (By J. YOUNG, B.A.)—If two parallel chords of two concentric circles are at a given distance apart, prove that the difference of their lengths is a minimum when they subtend the same angle at the centre. [Suggested by Quest. 6605, *Reprint*, Vol. XXXV., p. 99, whereof the property now enunciated furnishes an easy and rigorous proof.]

Solution by MARY S. MEYER; C. MORGAN, B.A.; and others.

Let AB, CD be the two parallel chords at the given distance, and subtending same angle at the common centre, draw OCA and ODB, draw EF parallel to AB and very close to it, draw OE, and through G, where OE cuts the inner circle, draw GH parallel to AB; then CG is parallel to AE, and smaller. Produce CG to K till CK = AE, and draw KLM parallel to AB, cutting the inner circle in L and M; then, as AE is parallel and = CK, CP, the distance of CD and LM = AN, that of AB and EF, and therefore the chords EF, LM are at the given distance, and EN = KP. But, in passing from the chords AB, CD,



to EF, LM, one chord is increased by 2EN, and the other by the smaller quantity 2LP; the difference, therefore, in the case of the first pair is less than in the case of the consecutive pair, and the same follows if EF be drawn above AB; the difference of the pair AB and CD is therefore minimum.

[If AB, CD be the two chords in any position, and CT be drawn perpendicular to AB, we have $CA^2 = AT^2 + CT^2$; but CT is given in magnitude, therefore AT is least when CA is least, which is the case when CA passes through the common centre.]

6950. (By C. LEUDESDOFF, M.A.)—Prove that, if

$$a [b^2x^2 + c^2y^2 + yz (b^2 + c^2 - a^2)]^{\frac{1}{2}} + b [c^2x^2 + a^2z^2 + zx (c^2 + a^2 - b^2)]^{\frac{1}{2}} \\ + c [a^2y^2 + b^2x^2 + xy (a^2 + b^2 - c^2)]^{\frac{1}{2}} = 0,$$

and $x + y + z = 1$, then either $a^2yz + b^2zx + c^2xy = 0$ or $a \pm b \pm c = 0$.

Solution by the Rev. T. R. TERRY, F.R.A.S.; E. RUTTER; and others.

$$a^2 (b^2 + c^2 - a^2) yz + b^2 (c^2 + a^2 - b^2) zx - c^2 (a^2 + b^2 - c^2) xy - 2a^2b^2c^2 \\ = -2ab \{ b^2x^2 + c^2y^2 + yz (b^2 + c^2 - a^2) \}^{\frac{1}{2}} \{ c^2x^2 + a^2z^2 + xy (c^2 + a^2 - b^2) \}^{\frac{1}{2}}, \\ (a^2yz + b^2zx + c^2xy)^2 (a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2) = 0, \text{ therefore, \&c.}$$

5244. (By Professor CAYLEY, F.R.S.)—Writing for shortness

$$L = \beta^2\gamma^3 - a^2\delta^2, \quad M = \gamma^2a^2 - \beta^2\delta^2, \quad N = a^2\beta^2 - \gamma^2\delta^2,$$

$$F = a^2 + \delta^2 - \beta^2 - \gamma^2, \quad G = \beta^2 + \delta^2 - \gamma^2 - a^2, \quad H = \gamma^2 + \delta^2 - a^2 - \beta^2,$$

$$\Delta = + \beta^2 + \gamma^2 + \delta^2;$$

show that the equation

$$LMN (x^4 + y^4 + z^4 + w^4) + MN (F\Delta + 2L)(y^2z^2 + x^2w^2) \\ + NL (G\Delta + 2M)(x^2x^2 + y^2w^2) + LM (H\Delta + 2N)(x^2y^2 + z^2w^2) \\ - 2a\beta\gamma\delta FGH\Delta xyzw = 0$$

belongs to a 16-nodal quartic surface, having the nodes

$$\begin{array}{l} x = a \left| \begin{array}{cccc|cccc|cccc} a & \beta & \beta & \beta & \gamma & \gamma & \gamma & \delta & \delta & \delta \\ \beta & -\beta & -\beta & \beta & a & -a & -a & a & \delta & -\delta & -\delta \\ \gamma & -\gamma & \gamma & -\gamma & \delta & -\delta & \delta & -\delta & a & -a & a \\ \delta & \delta & -\delta & -\delta & \gamma & \gamma & -\gamma & -\gamma & \beta & -\beta & -\beta \end{array} \right| \\ y = \beta \left| \begin{array}{cccc|cccc|cccc} a & \beta & \beta & \beta & \gamma & \gamma & \gamma & \delta & \delta & \delta \\ \beta & -\beta & -\beta & \beta & a & -a & -a & a & \delta & -\delta & -\delta \\ \gamma & -\gamma & \gamma & -\gamma & \delta & -\delta & \delta & -\delta & a & -a & a \\ \delta & \delta & -\delta & -\delta & \gamma & \gamma & -\gamma & -\gamma & \beta & -\beta & -\beta \end{array} \right| \\ z = \gamma \left| \begin{array}{cccc|cccc|cccc} a & \beta & \beta & \beta & \gamma & \gamma & \gamma & \delta & \delta & \delta \\ \beta & -\beta & -\beta & \beta & a & -a & -a & a & \delta & -\delta & -\delta \\ \gamma & -\gamma & \gamma & -\gamma & \delta & -\delta & \delta & -\delta & a & -a & a \\ \delta & \delta & -\delta & -\delta & \gamma & \gamma & -\gamma & -\gamma & \beta & -\beta & -\beta \end{array} \right| \\ w = \delta \left| \begin{array}{cccc|cccc|cccc} a & \beta & \beta & \beta & \gamma & \gamma & \gamma & \delta & \delta & \delta \\ \beta & -\beta & -\beta & \beta & a & -a & -a & a & \delta & -\delta & -\delta \\ \gamma & -\gamma & \gamma & -\gamma & \delta & -\delta & \delta & -\delta & a & -a & a \\ \delta & \delta & -\delta & -\delta & \gamma & \gamma & -\gamma & -\gamma & \beta & -\beta & -\beta \end{array} \right| \end{array}$$

and the sixteen singular tangents planes represented by the equations $(a, \beta, \gamma, \delta) (x, y, z, w) = 0$, &c.

Solution by J. HAMMOND, M.A.; G. EASTWOOD, M.A.; and others.

From the conditions of the question we have obviously

$$\left. \begin{aligned} 4\alpha^2 &= \Delta + F - G - H \\ 4\beta^2 &= \Delta + G - H - F \\ 4\gamma^2 &= \Delta + H - F - G \\ 4\delta^2 &= F + G + H + \Delta \end{aligned} \right\}, \quad \left. \begin{aligned} 4L &= GH - F\Delta \\ 4M &= HF - G\Delta \\ 4N &= FG - H\Delta \end{aligned} \right\} \dots\dots\dots(1, 2),$$

$$\left. \begin{aligned} 4(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) &= F^2 + G^2 + H^2 + \Delta^2 \\ 2(\delta^4 + \alpha^4 - \beta^4 - \gamma^4) &= GH + F\Delta \\ 2(\delta^4 + \beta^4 - \gamma^4 - \alpha^4) &= HF + G\Delta \\ 2(\delta^4 + \gamma^4 - \alpha^4 - \beta^4) &= FG + H\Delta \end{aligned} \right\} \dots\dots\dots(3),$$

$$\left. \begin{aligned} \Delta^2 + F^2 - G^2 - H^2 &= 8(\beta^2\gamma^2 + \alpha^2\delta^2) \\ \Delta^2 + G^2 - H^2 - F^2 &= 8(\gamma^2\alpha^2 + \beta^2\delta^2) \\ \Delta^2 + H^2 - F^2 - G^2 &= 8(\alpha^2\beta^2 + \gamma^2\delta^2) \end{aligned} \right\} \dots\dots\dots(4).$$

And if A, B, C, D denote the first minors of the determinant

$$\equiv A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 = FGH\Delta,$$

$$\left\{ \begin{array}{l} \alpha^2, \beta^2, \gamma^2, \delta^2 \\ \beta^2, \alpha^2, \delta^2, \gamma^2 \\ \gamma^2, \delta^2, \alpha^2, \beta^2 \\ \delta^2, \gamma^2, \beta^2, \alpha^2 \end{array} \right\} \left\{ \begin{array}{l} A + B + C + D = FGH \\ D + A - B - C = GH\Delta \\ D + B - C - A = HF\Delta \\ D + C - A - B = FG\Delta \end{array} \right\} \dots\dots\dots(5);$$

the first of these relations being obtained by adding all the rows of the determinant and dividing by Δ , and the others in like manner. From (5) we have (6), whilst (2) and (3) with (5) readily give (7) and (8) :-

$$\left. \begin{aligned} F^2G^2H^2 + G^2H^2\Delta^2 + H^2F^2\Delta^2 + F^2G^2\Delta^2 &= 4(A^2 + B^2 + C^2 + D^2) \\ F^2G^2H^2 + G^2H^2\Delta^2 - H^2F^2\Delta^2 - F^2G^2\Delta^2 &= 8(BC + AD) \\ F^2G^2H^2 + H^2F^2\Delta^2 - F^2G^2\Delta^2 - G^2H^2\Delta^2 &= 8(CA + BD) \\ F^2G^2H^2 + F^2G^2\Delta^2 - G^2H^2\Delta^2 - H^2F^2\Delta^2 &= 8(AB + CD) \end{aligned} \right\} \dots\dots\dots(6),$$

$$\left. \begin{aligned} LFGH\Delta &= AD - BC \\ MFGH\Delta &= BD - CA \\ NFGH\Delta &= CD - AB \end{aligned} \right\} \dots\dots\dots(7),$$

$$\left. \begin{aligned} (\delta^4 + \alpha^4 - \beta^4 - \gamma^4) FGH\Delta &= D^2 + A^2 - B^2 - C^2 \\ (\delta^4 + \beta^4 - \gamma^4 - \alpha^4) FGH\Delta &= D^2 + B^2 - C^2 - A^2 \\ (\delta^4 + \gamma^4 - \alpha^4 - \beta^4) FGH\Delta &= D^2 + C^2 - A^2 - B^2 \end{aligned} \right\} \dots\dots\dots(8).$$

The manner of obtaining these is extremely simple; for instance,

$$\begin{aligned} 4LFGH\Delta &= FGH\Delta(GH - F\Delta) = FGH \cdot GH\Delta - HF\Delta \cdot FG\Delta \\ &= (A + D)^2 - (B + C)^2 - (D - A)^2 + (B - C)^2 = 4(AD - BC). \end{aligned}$$

If moreover we write $a = LMN$, $b = MN(F\Delta + 2L)$, $c = NL(G\Delta + 2M)$, $d = LM(H\Delta + 2N)$, expressing these in terms of F, G, H, Δ by means of (2), and then expanding and reducing by means of (6), (3), (4), we have

$$\left. \begin{aligned} 64a &= (GH - F\Delta)(HF - G\Delta)(FG - H\Delta) \\ 32b &= (GH + F\Delta)(HF - G\Delta)(FG - H\Delta) \\ 32c &= (GH - F\Delta)(HF + G\Delta)(FG - H\Delta) \\ 32d &= (GH - F\Delta)(HF - G\Delta)(FG + H\Delta) \end{aligned} \right\} \dots\dots\dots(9).$$

$$\left. \begin{aligned} 16a &= A^2 + B^2 + C^2 + D^2 - FGH\Delta (a^4 + \beta^4 + \gamma^4 + \delta^4) \\ 4b &= BC + AD + FGH\Delta (\beta^2\gamma^2 + a^2\delta^2) \\ 4c &= CA + BD + FGH\Delta (\gamma^2a^2 + \beta^2\delta^2) \\ 4d &= AB + CD + FGH\Delta (a^2\beta^2 + \gamma^2\delta^2) \end{aligned} \right\} \dots\dots\dots(10).$$

It is evident that if we consider the surface

$$\phi(x, y, z, w) \equiv a(x^4 + y^4 + z^4 + w^4) + b(y^2z^2 + z^2w^2) + c(z^2x^2 + x^2w^2) + d(x^2y^2 + y^2w^2) - 2a\beta\gamma\delta FGH\Delta xyzw = 0 \dots\dots\dots(11),$$

$$\begin{aligned} \phi(x, y, z, w) &= \phi(x, -y, -z, w) = \phi(x, -y, z, -w) = \phi(x, y, -z, -w) \\ &= \phi(y, x, w, z) = \phi(z, w, x, y) = \phi(w, z, y, x); \end{aligned}$$

and it will suffice to show that the point $(a, \beta, \gamma, \delta)$ is a node, and that the plane $(a, \beta, \gamma, \delta)$ is a singular tangent. Beginning with the tangent planes, with the relations (12) we have (13):—

$$\left. \begin{aligned} ax + \beta y + \gamma z + \delta w &= W \\ ax - \beta y - \gamma z + \delta w &= X \\ -ax + \beta y - \gamma z + \delta w &= Y \\ -ax - \beta y + \gamma z + \delta w &= Z \end{aligned} \right\} \dots\dots\dots(12),$$

$$\begin{aligned} XYZW &= \{ (ax + \delta w)^2 - (\beta y + \gamma z)^2 \} \{ (\delta w - ax)^2 - (\beta y - \gamma z)^2 \} \\ &= (a^2x^2 - \beta^2y^2 - \gamma^2z^2 + \delta^2w^2)^2 - 4(\beta\gamma yz - a\delta xw)^2 \\ &= a^4x^4 + \beta^4y^4 + \gamma^4z^4 + \delta^4w^4 - 2\beta^2\gamma^2y^2z^2 - 2\gamma^2a^2x^2w^2 - 2a^2\beta^2x^2y^2 \\ &\quad - 2a^2\delta^2x^2w^2 - 2\beta^2\delta^2y^2w^2 - 2\gamma^2\delta^2z^2w^2 + 8a\beta\gamma\delta xyzw \dots\dots\dots(13). \end{aligned}$$

$$\begin{aligned} \text{Hence } 4\phi(x, y, z, w) + FGH\Delta XYZW &= P_1x^4 + P_2y^4 + P_3z^4 + P_4w^4 \\ &\quad + 2Q_1y^2z^2 + 2Q_2x^2y^2 + 2Q_3x^2w^2 + 2R_1x^2w^2 + 2R_2y^2w^2 + 2R_3z^2w^2 \dots\dots(14), \end{aligned}$$

where

$$\begin{array}{l|l|l} P_1 = 4a + a^4 FGH\Delta & Q_1 = 2b - \beta^2\gamma^2 FGH\Delta & R_1 = 2b - a^2\delta^2 FGH\Delta \\ P_2 = 4a + \beta^4 FGH\Delta & Q_2 = 2c - \gamma^2a^2 FGH\Delta & R_2 = 2c - \beta^2\delta^2 FGH\Delta \\ P_3 = 4a + \gamma^4 FGH\Delta & Q_3 = 2d - a^2\beta^2 FGH\Delta & R_3 = 2d - \gamma^2\delta^2 FGH\Delta \\ P_4 = 4a + \delta^4 FGH\Delta & & \end{array}$$

Now by (10) and (8) we have (15), and the last three equations in (10) may be written in the form (16):—

$$\left. \begin{aligned} P_1 + P_2 + P_3 + P_4 &= A^2 + B^2 + C^2 + D^2 \\ P_4 + P_1 - P_2 - P_3 &= D^2 + A^2 - B^2 - C^2 \\ P_4 + P_2 - P_3 - P_1 &= D^2 + B^2 - C^2 - A^2 \\ P_4 + P_3 - P_1 - P_2 &= D^2 + C^2 - A^2 - B^2 \end{aligned} \right\} \begin{aligned} Q_1 + R_1 &= BC + AD \\ Q_2 + R_2 &= CA + BD \\ Q_3 + R_3 &= AB + CD \end{aligned} \dots\dots(15, 16);$$

and, noticing that $R_1 - Q_1 = LFGH\Delta$ &c., equations (7) give

$$\left. \begin{aligned} R_1 - Q_1 &= AD - BC \\ R_2 - Q_2 &= BD - CA \\ R_3 - Q_3 &= CD - AB \end{aligned} \right\} \dots\dots\dots(17).$$

The equation to the surface is now reduced by means of (14) to the form

$$(Ax^2 + By^2 + Cz^2 + Dw^2)^2 = FGH\Delta XYZW \dots\dots\dots(18),$$

showing that $X, Y, Z, W = 0$ are four singular tangents, and that their conics of contact all lie on the same quadric, viz.,

$$Ax^2 + By^2 + Cz^2 + Dw^2 = 0.$$

It may be noticed that, when $x = \alpha$, $y = \beta$, $z = \gamma$, $w = \delta$, both sides of (18) at once reduce to $F^2G \cdot H^2\Delta^2$, and that for $x = \beta$, $y = \alpha$, $z = \delta$, $w = \gamma$, both sides vanish. Returning to (11), we have identically

$$\begin{aligned} z \frac{d\phi}{dx} + y \frac{d\phi}{dy} + z \frac{d\phi}{dz} + w \frac{d\phi}{dw} &= 4\phi(x, y, z, w) \\ z \frac{d\phi}{dx} - y \frac{d\phi}{dy} - z \frac{d\phi}{dz} + w \frac{d\phi}{dw} &= 4a(x^4 - y^4 - z^4 + w^4) - 4b(y^2z^2 - x^2w^2) \\ -z \frac{d\phi}{dx} + y \frac{d\phi}{dy} - z \frac{d\phi}{dz} + w \frac{d\phi}{dw} &= 4a(y^4 - x^4 - z^4 + w^4) - 4c(x^2z^2 - y^2w^2) \\ -x \frac{d\phi}{dx} - y \frac{d\phi}{dy} + z \frac{d\phi}{dz} + w \frac{d\phi}{dw} &= 4a(x^4 - x^4 - y^4 + w^4) - 4d(x^2y^2 - z^2w^2), \end{aligned}$$

so that, if we make $x = \alpha$, $y = \beta$, $z = \gamma$, $w = \delta$, these expressions are, by (2) and (3), reduced to

$$\begin{aligned} 2a(GH + F\Delta) - b(GH - F\Delta), \quad 2a(HF + G\Delta) - c(HF - G\Delta), \\ 2a(FG + H\Delta) - d(FG - H\Delta), \end{aligned}$$

which are seen, by (9), to vanish identically. Hence, by addition and subtraction, $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$, and $\frac{d\phi}{dw}$ all of them vanish at the point $(\alpha, \beta, \gamma, \delta)$, which is therefore a node.

[For many beautiful properties of this surface reference may be made to the following papers:—

BORCHARDT—*Ueber die Darstellung der Kummer'sche Fläche*, &c., Crelle, t. 83, p. 234. 1877;

CAYLEY—*On the Double Θ -Functions in connexion with a 16-nodal Quartic Surface*, Crelle, t. 83, p. 210; *On the 16-nodal Quartic Surface*, Crelle, t. 84, p. 238;

KUMMER—*Ueber die Flächen vierten Grades, mit sechzehn singulären Punkten*, Monatsberichte der Berlin Akad. 1864, p. 246;

WEBER—*Ueber die Kummer'sche Fläche*, &c.; Crelle, t. 84, p. 232.]

PROBABILITY AND LISTERISM. No. 2. By the Editor.

As a discussion bearing on the principles of the Theory of Probability is always useful and interesting, we propose to insert here, in connexion with Quest. 6929, some further remarks and solutions, selected from the correspondence we have received since the publication of our former article.*

1. Mr. WHITWORTH sends the following solution and remarks:—

“Let p be the probability of success under the old treatment, and μp the probability of success under the new (or LISTER'S) treatment. Then the *a priori* probability of the observed event was

$$P = p^9(1-p)^5(\mu p)^7(1-\mu p)^3,$$

and μ may have any value between 0 and p^{-1} . If we are asked what is the chance that $\mu = 1$, the answer is obviously zero, for the number of possible values that μ could have between the above-named limits is indefinitely large, and the chance of any particular one is indefinitely small. But if we are asked what is the chance that $\mu < 1$ (that is, that LISTER'S

* See pp. 59—61 of this Volume.

4. Dr. MACFARLANE sends the following additional remarks:—

"Dr. MACALISTER in his note says,—'I am only justified, perhaps, in saying that, if I were told it was great odds that a patient recovering under Listerism would have equally recovered without it, I should, knowing nothing more, have inevitably inferred that there were great odds against there being anything in Listerism.' And this is represented as my result. It is not only not my result, it is a conclusion which I would inevitably not make. If I were told the odds were 1 to 0 that a patient recovering under Listerism would have equally recovered without it, I should inevitably infer that there was nothing (either beneficial or injurious) in Listerism; but if I am told the odds were 45 to 4, then I inevitably infer that there certainly is something in Listerism, the something amounting to this that every twelfth patient recovering under that method would not have recovered under the ordinary method. Suppose that statistics are taken through a certain interval of time of the cases treated by a number of surgeons practising the ordinary method, and also through an equal interval of time of the cases treated by an equal number of equally skilful surgeons practising the Listerian method, and that under the former method the ratio of the successful cases to the whole number of cases is found to be q , and under the latter method p . The two methods have certain operations in common, the former has (or might have) some which the latter has not, and the latter has some which the former has not. The difference $p - q$ measures the value of leaving out those operations in the former method which are not in the latter, and introducing those in the latter which are not in the former. I am not aware that this principle has been enunciated before, and probably that is the reason why my solution has been regarded as somewhat obscure. I assumed, not without reason, that the data given were intended to give the actual values of p and q . In my previous note I ventured to say that Dr. MACALISTER's question was not a particularly clear way of asking what are the odds that the apparent success of LISTER's method was a real success; he replies that he did not mean to ask this. Let us have dry light from the balls and urns upon this matter. With respect to them he puts his question in the form, 'What are the odds that the apparent preponderance of white balls in A over those in B is not due to chance?' Further on we have, 'an expression giving the odds against a real preponderance of white balls in B;' and, still further on, the white balls are said to correspond to the successes. It is my opinion that, putting all these expressions together, we get something remarkably like that form into which I put the question; they at least were the grounds I went upon. By the apparent success of LISTER's method I simply mean the apparently greater ratio of successful cases as indicated by the data. Mr. MACCOLL does not misrepresent my solution, but says that he does not understand it. The above explanation will, I hope, make its meaning clear, and show that it neither contains discrepancy in its reasoning, nor leads, as Miss BLACKWOOD supposes, to a perplexing paradox. Grant her hypothesis of unbroken success through numberless trials on the side of LISTER balanced against a success of only 60 per cent. on the side of ordinary dressings, then my solution gives the very unparadoxical result that the measure of the value of the innovations embodied in Listerism is 40 per cent., or, to state the same thing in a different form, that it would be 2 to 3 that a person treated by that method owed his recovery to the innovations, or, as I termed them formerly, the characteristic part of the method."

whatever values between the limits 1 and n we give to m and r . If Dr. MACALISTER accepts the foregoing expanded statement as an accurate definition of the shorter, what, may I ask, has our knowledge or ignorance, as to whether the sequence of A by B will take place on any particular occasion, to do with the matter? Even an 'omniscient observer' might, consistently with this definition, assert that the chance of the sequence was $\frac{1}{2}$, as his fore-knowledge would be irrelevant to the question; though, of course, to such an observer the whole theory of probability would be a useless collection of truisms. In assuming the chance that LISTER's method has an advantage over ordinary methods to be $\frac{1}{2}$, Dr. MACALISTER virtually assumes that out of all the methods hitherto proposed by LISTER, or which may yet be proposed by him, some will have more success, and just as many less success, than the ordinary methods; and this assumption does not seem to me to be fairly implied in the data of his problem. He admits that the independent or *a priori* chance is a very important factor *when you know it*, but argues—on what grounds it is not easy to see—that we ought always to consider it $\frac{1}{2}$ *when we don't know it*. I am afraid I misunderstand Dr. MACALISTER's meaning here; for he surely does not advocate the adoption of a *probably false assumption* in order to obtain a *right conclusion*? Would it not be better to denote the *a priori* chance by some symbol a , find the required chance in terms of a , and wait patiently for further knowledge before we committed ourselves to any definite numerical result? Even in the case of Dr. MACALISTER's urns, such questions as, who placed the balls in the urns, and was he likely to place a larger proportion of white balls in the one than in the other, would introduce a disturbing element of which it would be well to take symbolic notice."

3. In reference to the preceding remarks, Miss BLACKWOOD writes as follows:—

"While agreeing in the main with Mr. MACCOLL's criticism of Dr. MACALISTER's views, I am afraid that although, like HUDBRAS,

'He is in logic a great critic, profoundly skilled in analytic,

And can distinguish and divide a hair 'twixt south and south-west side,'

he has himself, in attempting a precise and formal definition of the word *chance*, exposed a vulnerable spot, upon which his adversary (if he has a grain of logic in his mathematical composition) will strike with fatal effect. I await the *dénouement* with much curiosity. As to the right which Dr. MACALISTER claims to substitute, if he chooses, MUMBO JUMBO for LISTER in his solution, I am inclined to think he has already exercised that right. But, granting this mathematical licence of substitution 'for algebraical purposes' (a euphemism apparently for *juggling* purposes), is LISTER to be arbitrarily valued at $\frac{1}{2}$ merely *because we don't know what other value to assign to him*? Poor LISTER! I have to apologise to Dr. MACFARLANE for having unintentionally done him an injustice. My criticism in last month's *Educational Times* bears upon his *first* solution, as I understood it, and not upon his subsequent interpretation thereof—an interpretation, however, which, I cannot help agreeing with Dr. MACALISTER in thinking somewhat forced and misty."

In explanation of Miss BLACKWOOD's Hudibrastic allusion to Mr. MACCOLL's skill in "logic and in analytic," our readers are referred to his articles in the *Educational Times*, in the *Proceedings* of the Mathematical Society, in *Mind*, and in the *Transactions* of the Manchester Philosophical Society, and to his controversies with Prof. JEVONS and Mr. VENN in the *Philosophical Magazine* and in *Nature*.

NOTE ON EGYPTIAN ARITHMETIC. By T. MUIR, F.R.S.E.

In regard to the inquiry as to Egyptian Arithmetic [which arose out of Prof. SYLVESTER'S Quest. 6919, solved on pp. 42, 43 of this Volume] I would remark that the source of our knowledge of the said Arithmetic is mainly the *Rhind papyrus*, translated some years ago by *Eisenlohr*. New interest has been raised in connection with the subject by the recent publication of CANTOR'S *Lectures on the History of Mathematics*, which furnish perhaps all the information obtainable.

6839. (By Professor HUDSON, M.A.)—If α, δ be one pair of coordinates, l, λ another, α, l being measured from the same origin, and ω the inclination of the planes of reference, prove that

$$\cos \omega = \frac{\sin \lambda \tan l + \sin \delta \tan \alpha}{\sin \lambda \tan \alpha + \sin \delta \tan l}$$

Solution by A. McMURCHY, B.A.; BELLE EASTON; and others.

Let S be the place of a star; P, K the poles of equator and ecliptic; then, applying the formula $\cot A \sin B = \cot a \sin c - \cos c \cos B$ to ΔKSP ; equating values of $\sin \omega$ thereby obtained, and solving for $\cos \omega$, we have

$$\begin{aligned} \cos \omega &= \frac{\tan l \tan \lambda \cos \alpha + \tan \alpha \tan \delta \cos l}{\tan \lambda \sin \alpha + \tan \delta \sin l} \\ &= \frac{\tan l \sin \lambda \left(\frac{\cos \alpha}{\cos \lambda} \right) + \sin \delta \tan \alpha \left(\frac{\cos l}{\cos \delta} \right)}{\tan \alpha \sin \lambda \left(\frac{\cos \alpha}{\cos \lambda} \right) + \sin \delta \tan l \left(\frac{\cos l}{\cos \delta} \right)}. \end{aligned}$$

But $\cos \alpha \cos \delta = \cos l \cos \lambda$; hence striking out (), the result follows.

5397. (By ELIZABETH BLACKWOOD.)—If we take n positive random quantities $x_1, x_2, x_3, \dots, x_n$ between 0 and a , but with the *a priori* knowledge that all values of x_r between 0 and a^r are equally likely to be taken; show that the chance that x_m is the greatest of the n quantities is $2m \div n(n+1)$. [If we take $n = 3$, and make m successively = 3, 2, 1, we get the three answers to Quest. 5367, otherwise solved on p. 39 of Vol. XXIX. of the *Reprints*.]

Solution by J. HAMMOND, M.A.; H. MCCOLL, B.A.; and others.

Taking $x_r = y_r$, the chance will be

$$\int dy_m dy_1 dy_2 \dots dy_n + \int dy_1 dy_2 dy_3 \dots dy_n,$$

where, in the denominator, the limits of integration being 0 to a , 0 to a^2 , 0 to a^3 , ... 0 to a^n , we have $a^{1+2+3+\dots+n} = a^{i^n(n+1)}$ = denominator. But, in the numerator, the limits of y_k are 0 to x_m^k , k being different from m , and $y_m = 0$ to a^m , so that the chance is

$$\frac{1}{a^{i^n(n+1)}} \int_0^a x_m^{i^n(n+1)-m} dy_m = \frac{1}{a^{i^n(n+1)}} \int_0^a y_m^{\frac{n(n+1)}{2m}-1} dy_m = \frac{2m}{n(n+1)}.$$

6984. (By the EDITOR.)—Prove geometrically that the centre of one of the circles of contact (inscribed or escribed circles) of a triangle, the mid-point of one of the sides, and the mid-point of the line joining the opposite vertex with the point of contact of the circle with that side, are three points in the same straight line.

Solutions by (1) S. CONSTABLE; (2) M. BAKER; and others.

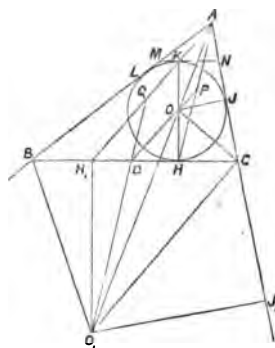
1. Let O be the centre of the inscribed circle; O_1 the centre of the escribed circle opposite the angle A ; and let H, H_1 , and J, J_1 be the points of contact of these circles with the sides BC, CA , respectively. Draw the diameter HOK of the circle (O), and the tangent MKN thereto parallel to BC ; and draw the other lines as in the diagram. Then we have

$$AO_1 : AO = O_1J_1 : OJ = O_1H_1 : OK;$$

therefore (Euc. vi. 4) the points A, K, H_1 are in the same straight line [as follows otherwise from the fact that K is the point of contact with the base MN of the triangle AMN of the circle (O) escribed to that side, so that AK produced must pass through the analogous point H_1 of the triangle ABC]; consequently the middle points (P, O, D) of AB, KH, HH_1 are in the same straight line parallel to AKH_1 . But the middle point of HH_1 is the middle point of the base, since $BH_1 = CH$; hence the theorem is proved for the *inscribed* circle (O).

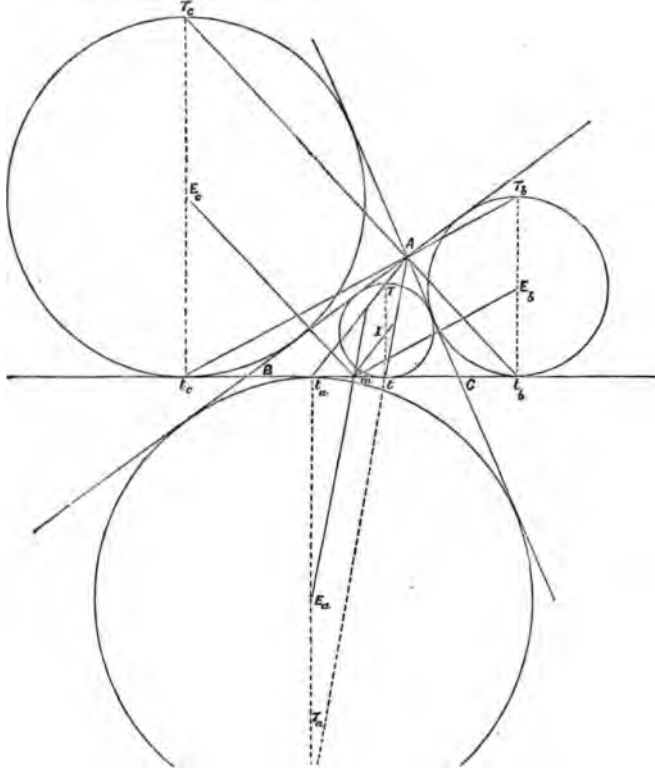
In like manner may we prove the analogous theorem for the *escribed* circle (O_1); viz., that the line joining the centre O_1 with the middle point D of the base BC bisects in Q the line AH_1 that joins the opposite vertex A with the point H_1 where the escribed circle touches the base.

[This theorem arose out of the solution of Quest. 6572, which appears on p. 110 of Vol. XXXVI. of the *Reprints*, having been made use of in that proof and verified by trilinear coordinates in an editorial *Note* affixed thereto. After these proofs were in type, the EDITOR found that, for the *inscribed* circle only, the theorem had been long since given in that wonderful repertory of geometrical questions, the *Diaries* (wherein many another supposed new theorem has been found), having, in fact, formed Quest. 812 of *Gentlemen's Diary* (for 1801) and Quest. 1138 of the *Ladies' Diary* (for 1805). Mr. YOUNG remarks that the parallelism of POD with



AKH_1 is a particular case of the theorem in geometrical conics that the line drawn parallel to the axis through the point of intersection of normals at the ends of a focal chord bisects the chord.]

2. *Otherwise*.—Draw the inscribed and escribed circles as in the annexed diagram, and join the vertical angle A to the points t, t_a, t_b, t_c ,



where the four circles of contact touch the side a . Then A is the direct centre of similitude of the circles I and E_a , and also the inverse centre of similitude of the circles E_b, E_c ; therefore τ and t_a are homologous points, as also are t and τ_a ; likewise t_b and τ_c are anti-homologous points, as are also t_c and τ_b . From this it follows that t and τ are the ends of a diameter; so also are t_a and τ_a ; t_b and τ_b ; t_c and τ_c . Now we have

$$Bt_c = Ct_b = s - a = s_1, \text{ and } Bt_a = Ct = s_3;$$

therefore

$$mt = mt_a \text{ and } mt_b = mt_c;$$

therefore a line from m , the middle point of tt_a , to I , the middle point of $t\tau$, is parallel to At_a , and therefore bisects At . Similarly $E_a m$ bisects At_a , $E_b m$ bisects At_b , and $E_c m$ bisects At_c .

6995. (By E. W. SYMONS, M.A.)—Given a, b, c all positive, and

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 1,$$

prove that the least value of $(a+1)(b+1)(c+1)$ is $\frac{27}{8}$.

Solution by G. HEFFEL, M.A.; R. TUCKER, M.A.; and others.

Let $a+1 = u$, $b+1 = v$, $c+1 = w$; then we have

$$\frac{u-1}{u} + \frac{v-1}{v} + \frac{w-1}{w} = 1, \text{ and } \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 2;$$

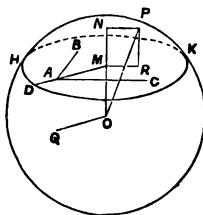
but $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} > \frac{3}{\sqrt[3]{uvw}}$, hence $uvw > \frac{27}{8}$, and the minimum value, when $a = b = c = \frac{1}{2}$, is $\frac{27}{8}$.

6996. (By Professor CAVALLIN, M.A.)—If p be the perpendicular from the curve of a sphere on the plane determined by three random points within the sphere; prove that the mean value of p^n is

$$\frac{1 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{(n+1)(n+6)(n+7)(n+8)(n+9)(n+10)} (2r)^n.$$

Solution by Prof. SEITZ, M.A.; CHRISTINE LADD, B.A.; and others.

Let DKH be the section of the sphere made by the plane through A, B, C, three random points within the sphere; M its centre; O the centre of the sphere; OQ a radius such that AB is parallel to the plane MOQ; P any point in the surface of the sphere; and PR the perpendicular on DHK. Draw PN perpendicular to OM, or OM produced, and let OD = r , PR = p , AM = x , AB = y , AC = z , MD = x' , $\angle DOM = \theta$, $\angle MAB = \phi$, $\angle CAM = \psi$, $\angle MOQ = \mu$, the angle the plane MOQ



makes with a fixed plane through OQ = ω , $\angle PON = \nu$, and p^n is the mean value of p^n for given positions of A, B, C. Then $x' = r \sin \theta$, $p = r(\cos \nu - \cos \theta)$, if $\nu < \theta$, and $p = r(\cos \theta - \cos \nu)$, if $\nu > \theta$; an element of the sphere at A is $r \sin \theta d\theta \cdot 2\pi x dx$, at B it is $y^2 dy d\phi d\mu$, at C it is $\sin(\phi + \psi) \sin \mu x^2 dx d\psi d\omega$, and an element of the surface at P is $2\pi r^2 \sin \nu d\nu$. The limits of θ are 0 and $\frac{1}{2}\pi$; of x , 0 and x' , and tripled; of ϕ , $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ , $-\phi$ and $\frac{1}{2}\pi$, and doubled; of μ , 0 and π ; of ω , 0 and 2π ; of y , 0 and $2x \cos \phi = y'$; of z , 0 and $2x \cos \psi = z'$; and of ν , 0 and θ , and θ

and π . Hence we have

$$\begin{aligned} p_\theta^n &= \frac{1}{4\pi r^2} \int_0^\pi \frac{1}{4\pi r^2} \int_0^\pi r^n (\cos v - \cos \theta)^n \cdot 2\pi r^2 \sin v \, dv \\ &\quad + \frac{1}{4\pi r^2} \int_0^\pi r^n (\cos \theta - \cos v)^n \cdot \pi r^2 \sin v \, dv. \\ &= \frac{r^n}{2(n+1)} [(1 - \cos \theta)^{n+1} + (1 + \cos \theta)^{n+1}] \\ &= \frac{(2r)^n}{n+1} [(\sin \tfrac{1}{2}\theta)^{2n+2} + (\cos \tfrac{1}{2}\theta)^{2n+2}]. \end{aligned}$$

The whole number of ways the three points can be taken is $(\frac{1}{2}\pi r^2)^3$; therefore the mean value of p^n is

$$\begin{aligned} &\frac{6 \cdot 3^3}{4^3 \pi^2 r^6} \int_0^{1\pi} \int_0^{x'} \int_{-1\pi}^{+1\pi} \int_{-\phi}^{1\pi} \int_0^{2\pi} \int_0^{x'} p_\theta^n \sin \theta \, d\theta \cdot 2\pi x \, dx \, d\phi \sin(\phi + \psi) \\ &\quad \times \sin \mu \, d\mu \, d\omega \, y^2 \, dy \, x^2 \, dx \\ &= \frac{36}{\pi^2 r^6} \int_0^{1\pi} \int_0^{x'} \int_{-1\pi}^{+1\pi} \int_{-\phi}^{1\pi} \int_0^{2\pi} p_\theta^n \sin \theta \sin(\phi + \psi) \, d\theta \, x^7 \, dx \cos^3 \phi \, d\phi \\ &\quad \times \cos^3 \psi \, d\psi \sin \mu \, d\mu \, d\omega \\ &= \frac{144}{\pi r^6} \int_0^{1\pi} \int_0^{x'} \int_{-1\pi}^{+1\pi} \int_{-\phi}^{1\pi} p_\theta^n \sin \theta \sin(\phi + \psi) \, d\theta \, x^7 \, dx \cos^3 \phi \, d\phi \cos^3 \psi \, d\psi \\ &= \frac{18}{\pi r^6} \int_0^{1\pi} \int_0^{x'} \int_{-1\pi}^{+1\pi} [3(\tfrac{1}{2}\pi + \phi) + 2 \cos \phi + \sin^2 \phi \cos \phi] p_\theta^n \sin \theta \, d\theta \, x^7 \, dx \cos^3 \phi \, d\phi \\ &= \frac{315}{16 r^6} \int_0^{1\pi} \int_0^{x'} p_\theta^n \sin \theta \, d\theta \, x^7 \, dx \\ &= \frac{1260 (2r)^n}{n+1} \int_0^{1\pi} [(\sin \tfrac{1}{2}\theta)^{2n+2} + (\cos \tfrac{1}{2}\theta)^{2n+2}] \sin^2 \tfrac{1}{2}\theta \cos^3 \tfrac{1}{2}\theta \, d\theta \\ &= \frac{1260 (2r)^n}{n+1} \left\{ \frac{1}{n+6} - \frac{4}{n+7} + \frac{6}{n+8} - \frac{4}{n+9} + \frac{1}{n+10} \right\} = \text{the result stated.} \end{aligned}$$

6918. (By A. McINTOSH, B.A.)—Two tangents are drawn at right angles to an ellipse, p is the perpendicular from the centre of the ellipse on the chord of contact, and p' is the perpendicular from the intersection of the tangents on the same line; prove that $\frac{1}{pp'} = \frac{1}{a^2} + \frac{1}{b^2}$. What corresponds to this in the case of a quadric surface?

Solution by MARY S. MEYER; W. H. BLYTHE, M.A.; and others.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be a quadric surface, (x_1, y_1, z_1) a point not on the surface; then its polar with respect to the quadric is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1$.

If p and p' be the perpendiculars from the origin and point on this plane,

$$\frac{1}{pp'} = \left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} \right) + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right);$$

if $\frac{1}{pp'} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$, we obtain (neglecting accents)

$$\frac{x^2}{a^2} \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{y^2}{b^2} \left(\frac{1}{a^2} + \frac{1}{c^2} \right) + \frac{z^2}{c^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

This gives the locus of the point in order that $\frac{1}{pp'} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

If we now make c infinite, and consider the point in the plane of x, y only, the section of the quadric becomes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and of the locus of x_1, y_1, z_1 , $\frac{x^2}{a^2 b^2} + \frac{y^2}{a^2 b^2} = \frac{1}{a^2} + \frac{1}{b^2}$, or the circle $x^2 + y^2 = a^2 + b^2$, from every point on which the tangents to the ellipse intersect at right angles, the value of $\frac{1}{pp'}$ becomes $\frac{1}{a^2} + \frac{1}{b^2}$.

NOTE ON QUESTION 1621. By the EDITOR.

In a paper by M. MANNHEIM, in the *Proceedings* of the Royal Society, for February 1882, "Sur les Surfaces Homofocales du second ordre," the following theorem is given as due to M. H. FAURE, from his "Recueil de Théorèmes relatifs aux Sections Coniques, Paris, 1867":—"From a point (A) on the outer of two confocal conics, tangents are drawn to the inner; if a third conic be drawn through A, having as foci the points of contact of the tangents, it will have contact of the third order with the outer conic." This theorem was given by Prof. CROFTON, as Quest. 1621, in the *Educational Times* for December 1864, with a more general one, and solutions to both questions appeared in the number for June 1865, and thereafter on p. 89 of Vol. III. of the *Reprints*.

6986. (By W. S. M'CAY, M.A.)—A lamina moves in its own plane, starting from any position and returning to the same position after one revolution; prove that (1) all straight lines that have enveloped curves of equal length (l) are tangents to the same circle in the lamina; (2) the length l is equal to the circumference of this circle; also (3) as the length l varies, the circles are concentric and have the same centre as KEMPE'S circles.

Solution by Professor TOWNSEND, F.R.S.

If L, M, N be any three concurrent lines of the lamina; a, b, c the sines of the three angles they determine in pairs; l, m, n the lengths of the three curves they envelope in the course of their revolution with the lamina; p, q, r the lengths of the three perpendiculars let fall on them in any position from any fixed origin in the plane of the motion; and $d\omega$ the elementary angle of rotation of all three in passing with the lamina to the consecutive position in the plane: then since, in every position of the lamina, $ap + bq + cr = 0$, therefore, in the course of its complete revolution,

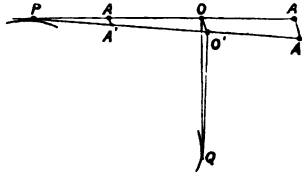
$\int_0^{2\pi} (ap + bq + cr) d\omega = 0$, and therefore, by LEGENDRE's theorem (WILLIAMSON's *Int. Cal.*, 3rd ed., p. 228), $al + bm + cn = 0$. Hence, if any two of the three lengths $l, m, n = 0$, the third also $= 0$; and therefore all lines of the lamina which envelope curves of evanescent length in the course of its revolution pass through a common point O of its area.

Again, if l and l' be the lengths of the two curves enveloped in the course of the revolution by any two parallel lines L and L' of the lamina, distant from each other by any interval k ; p and p' the lengths of the two perpendiculars let fall on them in any position from the origin; and $d\omega$, as before, their elementary angle of rotation corresponding to the position: then, since throughout the entire motion $p - p' = k$, therefore on its

completion $\int_0^{2\pi} (p - p') d\omega = 2\pi k$; and therefore by LEGENDRE's theorem, as above, $l - l' = s$, when $s = 2\pi k$. Hence, if $l' = 0$, that is, if L pass through O , then $l = s$; and therefore, all lines of the lamina which envelope curves of any common length l in the course of its revolution are tangents to a common circle of the same length l in the lamina, and the several circles for different values of l have a common centre O in its area.

To show that the point O of the lamina is that which, in the course of the revolution, describes the curve of least area in the plane of the motion, and that the areas of the curves described by all other points of it depend only on the distances of the describing points from O (WILLIAMSON's *Int. Cal.*, 3rd ed., p. 210, KEMPF's theorem).

If O and O' , see Fig., be any two consecutive positions of O , A and A' those of any other point A of the lamina, P the intersection of the two infinitely near lines OA and $O'A'$, $d\omega$ the elementary angle of displacement POO' of the lamina, a the constant length of the interval OA , regarded as positive throughout the entire motion for every position of A in the lamina, and p the variable length of the interval OP , regarded as positive or negative according as A and P lie at opposite sides or at the same side of O ; then, since manifestly, in the course of the complete revolution of the lamina, the excess (with its proper sign) of the area of the curve described by A over that of the curve described by O



$$= \frac{1}{2} \int_0^{2\pi} (AP^2 - OP^2) d\omega = \frac{1}{2} \int_0^{2\pi} (a^2 + 2ap) d\omega = \pi a^2 + a \int_0^{2\pi} p d\omega;$$

to prove the property, it remains only to show that in the course of the revolution $\int_0^{2\pi} p d\omega = 0$, which is readily done as follows:—

If Q, see Fig., be the intersection of the two infinitely near perpendiculars at O and O' to OA and O'A', q the length of the interval OQ, regarded as changing sign with every passage of Q through O in the course of the revolution, and ds the differential element at Q, intercepted between OQ and O'Q, of the curve enveloped by OQ in the plane of the motion; then since, by LEGENDRE'S theorem above, $ds - dq = p d\omega$, there-

fore, for the complete revolution of the lamina, $\int_0^{2\pi} p d\omega =$ the entire length

of the curve enveloped by OQ, which by the first part above = 0, since enveloped line passes through O. Hence, in the course of the revolution of the lamina, the area of the curve described by A exceeds that of the curve described by O by the area of the circle of which OA is radius.

On the several very elegant properties announced in the above question, many observations, as regards limitation and deduction, might be made with advantage. But the length of the solution, which could hardly be materially abridged, precludes the possibility of making any such here.

6982. (By Professor MINCHIN, M.A.)—Determine the velocity at every point of a moving liquid, as a function of the co-ordinates, in such a manner that at every point in the liquid there shall be screw motion.

Solution by J. J. WALKER, M.A.

Suppose the axis of z to be that of the screw, the components of the velocity of a particle, whose coordinates are xyz , being u, v, w ; then $x = r \cos \frac{2\pi}{h} z$, $y = r \sin \frac{2\pi}{h} z$, define the nature of the supposed movement, h being constant, throughout the liquid. Hence (r being constant for a given particle)

$$u = -\frac{2\pi}{h} r \sin \frac{2\pi z}{h} \cdot w = -\frac{2\pi}{h} y w, \quad v = \frac{2\pi}{h} r \cos \frac{2\pi z}{h} \cdot w = \frac{2\pi}{h} x w,$$

$$u^2 + v^2 = \frac{4\pi^2}{h^2} (x^2 + y^2) w^2.$$

Again, $\frac{du}{dx} = -\frac{2\pi}{h} r^{-2} x y w, \quad \frac{du}{dy} = \frac{2\pi}{h} r^{-2} x y w,$

so that the equation of continuity

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad \text{reduces to} \quad \frac{dw}{dz} = 0 \quad \text{or} \quad w = \phi(x, y).$$

The velocity V at any point is therefore given by

$$V^2 = u^2 + v^2 + w^2 = \left\{ 1 + \frac{4\pi^2}{h^2} (x^2 + y^2) \right\} [\phi(x, y)]^2.$$

[The velocity system $u = e^{\xi} \cos \left(\frac{2\pi z}{h} + \eta \right)$, $v = e^{\xi} \sin \left(\frac{2\pi z}{h} + \eta \right)$, $w = 0$,

where ξ and η are any two conjugate functions of x and y , will answer.]

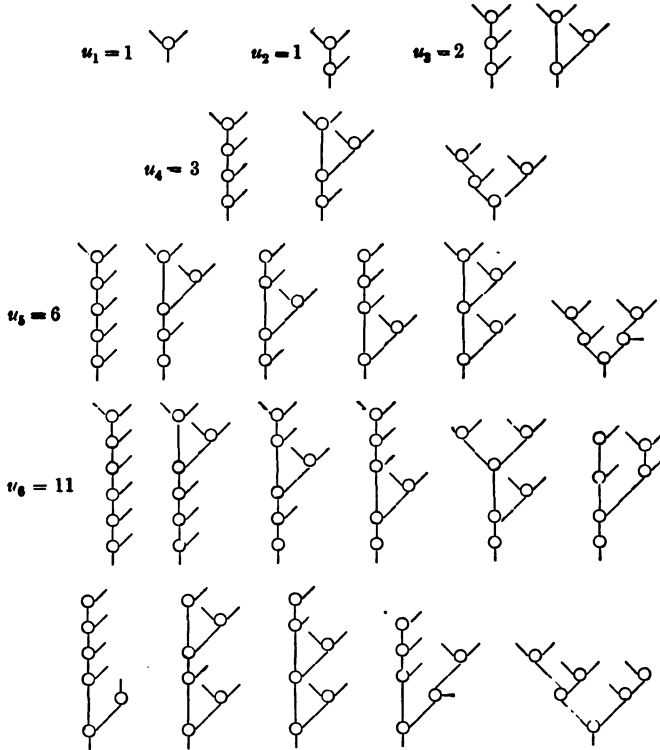
5688. (By Professor SYLVESTER, F.R.S.)—A tree proceeds from a fixed root, and is supposed subject to the law of bifurcation at each joint. Neglecting all distinctions founded on magnitude or direction, and putting u_n for the number of distinguishable forms of arborescence, when n is the number of joints, prove that, if

$$\phi x = -1 + x + u_1 x^2 + u_2 x^3 + u_3 x^4 + \dots, \text{ then } \phi(x^2) + (\phi x)^2 = -2x.$$

Solution by W. J. C. SHARP, M.A.; A. McMURCHY, B.A.; and others.

It is easy to show that $u_1 = 1, u_2 = 1, u_3 = u_2 + \frac{1}{2}(u_1^2 + u_1)$,

$$u_4 = u_3 + u_1 u_2, u_5 = u_4 + u_1 u_3 + \frac{1}{2}(u_2^2 + u_2), \text{ \&c. ;}$$



and the law thus indicated holds generally, so that

$$u_{2n} = u_{2n-1} + u_1 u_{2n-2} + u_2 u_{2n-3} + \dots + u_{n-1} u_n,$$

$$u_{2n+1} = u_{2n} + u_1 u_{2n-1} + u_2 u_{2n-2} + \dots + u_{n-1} u_{n+1} + \frac{1}{2}(u_n^2 + u_n).$$

For in the case of $n+1$ joints, and starting from the lowest joint, the remaining n joints may be arranged—

n in one branch in u_n ways,		and none in the other.	
$n-1$	„	u_{n-1}	1 „ in u_1 ways.
$n-2$	„	u_{n-2}	2 „ in u_2 ways.
&c.		&c.	

except that when n is even ($= 2p$) the different forms corresponding to two branches each with p joints will be the combinations of the u_p different forms (including repetitions) two together and will be $\frac{1}{2}(u_p^2 + u_p)$ in number;

$$2u_2x^3 = 2u_1x^3, \quad 2u_2x^4 = 2u_1x^4 + u_1^2x^4 + u_1x^4,$$

$$2u_4x^5 = 2u_2x^5 + 2u_1u_2x^5, \quad 2u_4x^6 = 2u_2x^6 + 2u_1u_2x^6 + u_2^2x^6 + u_2x^6, \quad \&c.,$$

therefore, by addition, $2 \{ \phi(x) + 1 - x - x^2 \}$

$$= \{ \phi(x) + 1 \}^2 - 2x^2 + \phi(x^2) + 1 \quad \text{or} \quad -2x = \{ \phi(x) \}^2 + \phi(x)^2.$$

6998. (By W. R. WESTROFF ROBERTS, M.A.) — Determine two independent conditions that the line, the coordinates of which are a, b, c, f, g, h , should lie in the plane $ax + \beta y + \gamma z + \delta w = 0$; and from these conditions show how to write down immediately the $n+1$ conditions involving the coordinates, that a line may lie in a surface of the n th degree, by writing its equation in a symbolical form.

Solution by H. W. L. TANNER, M.A.; CHRISTINE LADD, B.A.; and others.

The line whose coordinates are a, b, c, f, g, h is given by the intersection of any two of the planes whose equations are (1); and, expressing that $ax + \beta y + \gamma z + \delta w = 0$ passes through this intersection, we easily get the conditions (2):—

$$\begin{pmatrix} . & h, -g, a \\ -h, . & f, b \\ g, f, . & c \\ -a, -b, -c, . \end{pmatrix} (x, y, z, w) = 0, \quad \begin{pmatrix} . & c, -b, f \\ -c, . & a, g \\ b, -a, . & h \\ -h, -g, -h, . \end{pmatrix} (a, \beta, \gamma, \delta) = 0 \quad \dots (1, 2).$$

The second matrix is formed from the first by the substitution $(af)(bg)(ch)$. Of the conditions (2) two, and only two, are independent, since $af + bg + ch = 0$.

For the surface whose symbolical equation is $(ax + \beta y + \gamma z + \delta w)^n = 0$, the conditions are $A^{n-p} B^p = 0$ ($p = 0, 1, 2, \dots, n$), where A, B , are any pair of expressions on the left of (2). For instance, suppose the straight line given by the equations $gx - fy + cw = 0$, $ax + by + cz = 0$. Substituting for z, w from these in the equation of the surface, the latter becomes

$$\{ (ca - a\gamma - g\delta)x + (c\beta - b\gamma + f\delta)y \}^n = 0 \quad \text{or} \quad (-Bx + Ay)^n = 0.$$

This must be an identity (since otherwise it would determine a finite number of points in which the line meets the surface), and therefore the coefficients must vanish: that is, we must have

$$B^{n-p} \cdot A^p = 0, \quad (p = 0, 1, 2, \dots, n).$$

PROBABILITY AND LISTERISM. No. 3. By ELIZABETH BLACKWOOD.

Mr. WHITWORTH, in his solution of Quest. 6929, expresses the required chance as a function of a variable μ , and assumes that all values of this variable, within its limits of possibility, are equally probable. For this assumption I can discern no warrant whatever in the data of the question. Again, he remarks that, "if our knowledge is limited to the statement of the question, we must put $p = \frac{1}{14}$;" in other words, whenever we know that an event has happened 9 times out of 14, and know nothing else, we should conclude that the chance of its recurrence is $\frac{1}{14}$.

This, judging by experience, would be a treacherous principle to trust to; for it would lead us to infer that, when an event had happened three times running, the chance of its recurrence the fourth time would be $\frac{1}{4}$, or *certainty*. Now, I can assure Mr. WHITWORTH, as a positive fact, that I have myself actually seen a penny come down "heads" the first three tosses, and yet turn up "tails" at the fourth.

4626. (By Professor MALET, M.A.)—Given three conics passing through the same four points S, S' and S - kS', if through any point of S - kS' we draw two lines L and M; L intersecting S in the points a, b , and M intersecting S' in the points A, B; then, if the lines Aa and Bb intersect S' again in the points C and D, prove that the locus of the intersection of the lines (C, D) and (a, b) is the conic S - kS'.

Solution by the PROPOSER.

Let the second points of intersection of the lines aA and bB with the conic S be E and F, then for any position of the point and lines let the equations of Bb, Aa, ab, FE, AB, and CD be respectively $\alpha, \beta, \gamma, \delta, \lambda, \mu$; then the equations of S and S' must be respectively of the forms $\alpha\beta - k\gamma\delta$ and $\alpha\beta - k'\lambda\mu$. Subtracting, we see that the four points $(\gamma, \lambda), (\gamma, \mu), (\delta, \lambda), (\delta, \mu)$ lie on a conic passing through all the intersections of S and S', which proves the theorem.

6176, 6708, & 7009. (By the EDITOR.)—An iron hoop of radius t is thrown at random on (1) a round table of radius r , (2) a triangular table of inscribed radius r and semiperimeter s ; show that the probabilities (p_1, p_2) that the hoop will rest wholly on the table, and (q_1, q_2) that it will not fall off, are, respectively,

$$p_1 = \left(\frac{r-t}{r+t}\right)^2, \quad q_1 = \left(\frac{r}{r+t}\right)^2, \quad p_2 = \frac{(r-t)^2 s}{(r+2t)rs + \pi r t^2}, \quad q_2 = \frac{rs - \frac{1}{2}\pi t^2}{(r+2t)s + \pi t^2};$$

(3) if the hoop were a continuous disc (a florin for example) the probabilities would be the same with the omission of $-\frac{1}{2}\pi t^2$ in the numerator of q_2 ; and (4) if $t = r$ and the triangle in (2) be equilateral, the probabilities are

$$p_1 = 0, \quad q_1 = \frac{1}{4}, \quad p_2 = 0, \quad q_2 = \frac{3\sqrt{3}}{9\sqrt{3} + \pi} = .277.$$

Solution by Rev. T. R. TERRY, M.A.; W. H. BLYTHE, M.A.; and others.

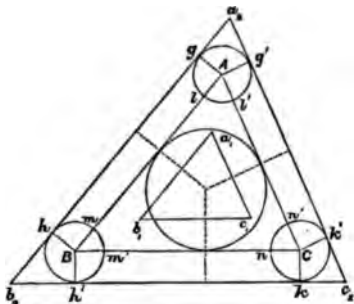
1. Since the hoop is thrown at random, all positions of its centre within a circle of radius $r+t$ are equally probable; hence, since circles are proportional to the squares of their radii, the chances p_1, q_1 are as stated.

2. Let ABC be the table; $a_1b_1c_1, a_2b_2c_2$ triangles whose sides are parallel to those of ABC at a distance t therefrom; and $gg'tl, hh'm'n, kk'n'n$ circles, each of radius t , around A, B, C as centres. Then, if the centre of the hoop fall anywhere within the mixtilineal space $gg'k'kh'h$, the hoop will touch the table; and if the hoop-centre fall within $a_1b_1c_1$, the hoop will rest wholly on the table; hence the probability that the hoop will rest wholly on the table is $p_2 = \text{area } a_1b_1c_1 + \text{area } gg'k'kh'h = \text{the result given in the question.}$

In order that the hoop may not fall off the table, two points at the ends of a diameter must rest on the table; it is not sufficient for the centre to fall within the triangle ABC (for if the centre were, for instance, at A , the hoop would evidently fall off); the centre of the hoop must, therefore, fall within $l'mm'n'n'$, where ll' is a circular arc from centre A with radius t , &c.; this area = $\text{area } ABC - \frac{1}{2}\pi t^2$, and the probability required is $(\text{area } ABC - \frac{1}{2}\pi t^2) + \text{area } gg'k'kh'h = \text{the result stated.}$

3. If the hoop were a continuous disc, so that it would rest if the centre were supported, the probability in the second case becomes that stated.

4. These values follow at once from (2) and (3).



7020. (By R. TUCKER, M.A.) — Prove that the envelop of the auxiliary circles of a system of ellipses, of equal major axes, whose foci lie on two fixed straight lines which intersect in a point on the system, is two parallel straight lines.

Solution by Prof. WOLSTENHOLME, M.A.; J. O'REGAN; and others.

Let S, H be the two foci, O the intersection of their loci; then, since $OS+OH$ is constant, the middle point of SH lies on a fixed straight line cutting off each of the given straight lines an intercept equal to half the given major axis. Hence we have a circle of constant radius (half the given major axis) with its centre on a fixed straight line; and it obviously has two fixed tangents, parallel to the locus of the centre and at distances from it each equal to half the given major axis.

It is perhaps rather a waste of time to do this analytically, but here it is: Taking the given lines as axes, let $2a$ be the given major axis, and let $2(a+\lambda), 2(a-\lambda)$ be the distances of the foci from the point common

to all the conics; the coordinates of the centre will be $a + \lambda$, $a - \lambda$, and the radius $2a$. Hence the equation will be, ω being the angle between the two fixed straight lines,

$$(x-a-\lambda)^2 + (y-a+\lambda)^2 + 2(x-a-\lambda)(y-a+\lambda)\cos\omega = 4a^2,$$

$$\text{or } (x-a)^2 + (y-a)^2 + 2(x-a)(y-a)\cos\omega - 4a^2 - 2\lambda(1-\cos\omega)(x-y) + 2\lambda^2(1-\cos\omega) = 0,$$

and writing X , Y for $x-a$, $y-a$, the envelop is

$$(X-Y)^2(1-\cos\omega) = 2(X^2 + Y^2 + 2XY\cos\omega - 4a^2),$$

which reduces to

$$(X+Y)^2 = \frac{8a^2}{1+\cos\omega}, \text{ or } X+Y = \pm 2a \sec \frac{1}{2}\omega; \text{ i.e., } x+y = 2a(1 \pm \sec \frac{1}{2}\omega).$$

6920. (By Professor TOWNSEND, F.R.S.)—Two forces in a common space being supposed to act along two non-intersecting right lines in the space; required the pitch of the equivalent wrench, in terms of the magnitudes of the forces, and of the distance and angle between their lines of direction in the space.

Solution by (1) G. S. CARR, B.A.; (2) *the PROPOSER.*

1. Let P , Q be the forces, α and δ their mutual inclination and distance, R the resultant along the central axis, G the resultant couple round it. It is known that $RG = PQ \cdot \delta \sin \alpha$. Also $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$. Therefore the pitch $G : R$ is determined in terms of P , Q , α and δ .

2. *Otherwise* :—Denoting by P and Q the two forces, by L and M their two lines of action in the space, by A and B the two extremities of the common perpendicular to L and M , by δ and α the distance AB and the angle LM , by R the resultant of P supposed transferred to B and of Q , by G the moment $P \cdot \delta$ resulting from the transference, by λ and μ the angles made by R with L and M , by ϕ the angle (evidently complementary to λ) made by R with the axis of G , and by x the required pitch; then, since, on well known elementary principles of force transference in a space,

$$R \cdot x = G \cdot \cos \phi = P \cdot \delta \cdot \sin \lambda = P \cdot \delta \cdot \frac{Q}{R} \cdot \sin \alpha;$$

therefore, at once,

$$x = \frac{P \cdot Q \cdot \delta \cdot \sin \alpha}{R^2} = \frac{P \cdot Q \cdot \delta \cdot \sin \alpha}{P^2 + Q^2 + 2PQ \cos \alpha},$$

which accordingly is the value expressed as required.

A parallel N to R through the point C on the line AB , which divides the distance AB in the inverse ratio of $P \cos \lambda : Q \cos \mu$, that is, in the inverse ratio of the projections of P and Q on R , is easily seen to be the axis of the wrench in question, to which the two forces P and Q , when reduced to their canonical form, are equivalent in their space.

6992. (By G. F. WALKER, M.A.)—Three masses m_1 , m_2 , m_3 are fastened to a string which passes through a ring, and m_1 describes a horizontal circle as a conical pendulum, while m_2 and m_3 hang vertically; if m_3 drop off, prove that the instantaneous change of tension of the string is $\frac{gm_1m_3}{m_1+m_2}$.

Solution by Prof. TOWNSEND, F.R.S.; J. O'REGAN; and others.

Denoting by θ the inclination of the conical pendulum to the vertical, and by S and T the tensions on the cord before and just after the separation of m_3 from the system; then, the accelerations of m_1 and m_2 along the cord being in both cases equal, and in the former case evanescent, we have consequently, for the determination of θ , of S , and of T , the three dynamical equations,

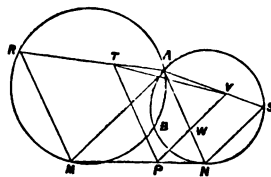
$$\frac{m_1 g \sec \theta - T}{m_1} = \frac{T - m_2 g}{m_2}, \quad \frac{m_1 g \sec \theta - S}{m_1} = \frac{S - (m_2 + m_3) g}{m_2 + m_3} = 0;$$

from which it appears, at once, that $m_1 \sec \theta = (m_2 + m_3)$, that $(m_1 + m_2) S = (m_1 + m_2)(m_2 + m_3)g$, that $(m_1 + m_2) T = m_1 m_2 (1 + \sec \theta)g$, and that consequently $(m_1 + m_2)(S - T) = m_1 m_2 g$, as stated in the question.

6958. (By Professor HADAMARD, M.A.)—Étant données deux circonférences O et O' qui se coupent aux points A et B , on fait tourner autour du point A un angle de grandeur constante, dont les côtés coupent les circonférences en deux points M et N . On divise la droite MN dans un rapport donné $m : n$ par le point P , et on demande le lieu de ce point.

Solution by (1) Dr. CURTIS; (2) W. S. MCCAY, M.A.

(1). Through M and N draw lines parallel respectively to AN and AM , and through P lines parallel to both; then, as $\angle ANS = \angle MAN$, and is therefore given, and A is fixed, S is also fixed, and also the line AS , and $AV : VS = AW : WN = MP : PN = m : n$; hence the point V is fixed, as is likewise the point T ; and, in the triangle TPV , the base TV is fixed, and the vertical angle $TPV (= \angle MAN)$ is given: the locus of P is therefore a circle.



(2). *Otherwise*:—Let the end of the vector ρ from a fixed point trace out a circle, then the extremity of the vector $\rho' \equiv q\rho + c$ will also trace out a circle, q being a constant quaternion in the plane and c a constant vector; for the operation q alters the lengths of all the vectors in a constant ratio and rotates them through a constant angle, and the operation c only translates the whole figure. And the second circle may be any circle by

proper choice of q and e . The corresponding points on the two circle are such that the central radii are inclined at a constant angle, and therefore the line joining corresponding points will subtend a constant angle at either intersection of the circles.

The vector from the origin to the point dividing the line in the ratio $m : n$ is $(np + mp') \div (m + n)$, which is of the form $q\rho + \rho'$; and the locus of the extremity is therefore a circle. It may be easily shown that there also exists a fixed point, the ratio of whose distances to corresponding points is constant.

7010. (By J. J. WALKER, M.A.)—Referring to the EDITOR'S Question 6984, show that, in the spherical triangle, the sines of the segments into which the arc joining the middle point of base with the centre of the inscribed (or escribed) circle divides the arc joining the vertex with the point of contact of that circle with the base, are as the cosines of the half sum of sides and of half the base.

Solution by the PROPOSER.

The vertex being B, O the centre of the circle, D its point of contact with CA, and E the point in which BO meets CA; then, considering the transversal arc MON (joining the centre M of the base CA with the centre O of the circle) as cutting the three sides of the triangle BDE in the points M, O, N,

$$\begin{aligned}\sin BN : \sin ND &= \sin BO : \sin OE \times \sin EM : \sin DM \\ &= \sin AB : \sin AE \times \sin \frac{1}{2}(CD - AD) : \sin \frac{1}{2}(c - a) \\ &= (\sin c - \sin a) : (\sin CD - \sin AD) \times \dots \\ &= \sin \frac{1}{2}(c - a) \cos \frac{1}{2}(c + a) : \sin \frac{1}{2}(CD - AD) \cos \frac{1}{2}(CD + AD) \times \dots \\ &= \cos \frac{1}{2}(c + a) : \cos \frac{1}{2}\delta.\end{aligned}$$

7024. (By J. MACLEOD, M.A.)—In the produced part of the diameter of a circle, find a point from which, if a tangent be drawn to the circle, it shall be n times the produced part of the diameter, n being any number, whole or fractional, greater than unity.

Solution by G. EASTWOOD, M.A. ; E. RUTTER ; and others.

Let AB be the given diameter ; in the tangent to the circle at B take a point Q such that BQ bears to the diameter BA the given ratio $1 : n$. Join AQ meeting the circle in P, and draw PT, the tangent at P, to meet AB in T ; then will $TP = n \cdot TB$.

7005. (By Prof. WOLSTENHOLME, M.A.)—Show in one figure the curves

- (1) $x^2 + ay = a^2$, (2) $y^2 + ay + ax = a^2$, (3) $x^2 - y^2 = ax$,
 (4) $y^2 - xy = a^2$, (5) $x^2 + y^2 + ax + 2ay = 2a^2$,
 (6) $x^2 - xy + y^2 + ay = 2a^2$, (7) $xy + ax + ay = 0$,
 (8) $x^3 + y^3 + 4axy + a^3 = 0$: which have in common the points
 (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , where

$$x_1 = -\cdot4450418a, \quad x_2 = 1\cdot2469796a, \quad x_3 = -1\cdot8019376a,$$

$$y_1 = \cdot8019376a, \quad y_2 = -\cdot5549582a, \quad y_3 = -2\cdot2469796a.$$

These points satisfy all the following equations:—

$$x_2 + y_2 = x_3 + y_1 = x_1 + y_2 = -a, \quad x_2y_2 = x_1y_3 = x_2y_1 = a^2,$$

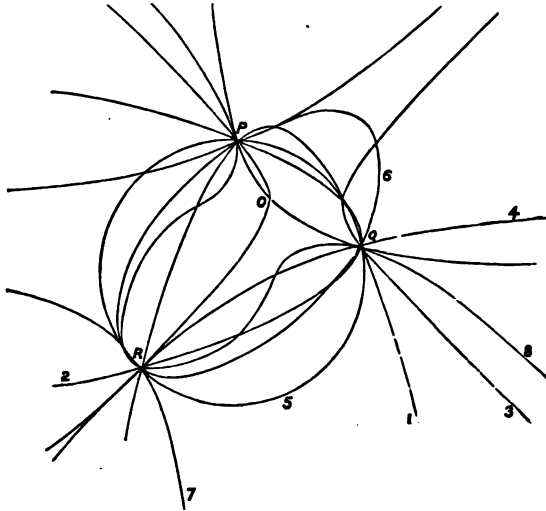
$$x_2x_3 + x_3x_1 + x_1x_2 = -2a^2, \quad x_1x_2x_3 = y_1y_2y_3 = a^3,$$

$$x_1 + x_2 + x_3 = -a, \quad (a + x_1)(a + x_2)(a + x_3) = -a^3,$$

$$y_1 + y_2 + y_3 = -2a, \quad (a + y_1)(a + y_2)(a + y_3) = -a^3, \text{ \&c. \&c.}$$

Solution by G. HERFEL, M.A.; Prof. MATZ, M.A.; and others.

In the figure, O is the origin, P, Q, R, are the three points, and the



curves are numbered as in the equations. (1), (2) are parabolas, (3), (4), and (7) are hyperbolas, (5) is a circle, (6) an ellipse, (8) a curve giving the outline of a vase, the lip on either side branching out, and having an asymptote inclined 45° to the axis of x . The equations may be written

- (1) $U = 0$; (2) $V = 0$; (3) $U - V = 0$; (4) $X = 0$;
 (5) $U + V = 0$; (6) $U + X = 0$; (7) $V - X = 0$;
 (8) $U(x - y - a) + V(x + y + a) - X(x + a) = 0$.

Now from U and V, and from U and X, we derive the same pairs of equations for x and y ; namely,

$$x^3 + ax^2 - 2a^2x - a^3 = 0, \text{ and } y^3 + 2ay^2 - a^2y - a^3 = 0.$$

Therefore the curves all pass through the three points thus determined.

Also, the two equations are so related that, if a is a root of the first, $\frac{1}{a}$ is a root of the second. And, if $-a-y$ be put for x in the first equation, we obtain the second; so that, if a is a root of the first, $-a-a$ is a root of the second. When one of the roots, as x_2 , is found by approximation, these relations at once give the rest.

The cubic curve can be most easily traced, by turning the axes through 45° , putting $a\sqrt{2} = c$, changing the origin to a point $x = \frac{2}{3}c$, then taking c as the unit of measurement the equation to the curve becomes

$$y^2 = -\frac{(2+x)^2}{3} - \frac{.5617}{x}.$$

[If, in the equation $x^3 + ax^2 - 2a^2x - a^3 = 0$, we put $x = a \cos \theta$, we get $2 \cos 3\theta + 2 \cos 2\theta + 2 \cos \theta + 1 = 0$; or, multiplying by $\sin \frac{1}{2}\theta$, $\sin \frac{1}{2}\theta = 0$, whence the three values of $\frac{x}{a}$ are $2 \cos \frac{2}{3}\pi$, $2 \cos \frac{4}{3}\pi$, $2 \cos \frac{8}{3}\pi$. The three points may then easily be proved to be

$$\frac{x_1}{-\sin a} = \frac{y_1}{\sin 2a} = \frac{a}{\sin 3a}, \quad \frac{x_2}{\sin 3a} = \frac{y_2}{-\sin a} = \frac{a}{\sin 2a},$$

$$\frac{x_3}{\sin 2a} = \frac{y_3}{\sin 3a} = \frac{a}{-\sin a}, \text{ where } 7a = \pi.$$

This suggests that probably the simplest form of a conic through three given points is

$$lX + mY + nZ = 0,$$

where $X \equiv x^2 + yz - y^2$, $Y \equiv y^2 + zx - x^2$, $Z \equiv x^2 + xy - x^2$.]

6978. (By Professor SYLVESTER, F.R.S.)—If i, j are any two positive integers and j not greater than i , prove that

$$(x-1)(x-2)\dots(x-i+1) - \frac{j}{i-j+1}x(x-1)\dots(x-i+2) + \frac{j(j-1)}{(i-j+1)(i-j+2)} \\ \times (x+1)x\dots(x-i+3) \\ \dots + (-)^j \frac{j(j-1)\dots 1}{(i-j+1)(i-j+2)\dots i} (x+j-1)(x+j-2)\dots(x+j-i+1) \\ = \frac{i-j}{i} \frac{(x-1)(x-2)\dots(x-i)}{(x+j-i)}.$$

$$\text{Ex.}-(x-1)(x-2)-\frac{1}{2}x(x-1) = \frac{2}{3} \frac{(x-1)(x-2)(x-3)}{x-2}. \quad [i=3, j=1.]$$

$$(x-1)(x-2)(x-3)-\frac{3}{2}x(x-1)(x-2) + \frac{2 \cdot 1}{3 \cdot 4}(x+1)x \cdot (x-1) \\ = \frac{2}{3} \frac{(x-1)(x-2)(x-3)(x-4)}{x-2}. \quad [i=4, j=2.]$$

[Prof. SYLVESTER states that the above singular formula occurred to him in the course of investigating a special case of his general theorem "On the extraction of the roots of any linear substitution," which is written out ready for publication in the *American Journal of Mathematics*.]

Solutions by (1) G. HEFFEL, M.A.; (2) Rev. T. R. TERRY, M.A.

(1). If we put $j = 1$, the series reduces to two terms, and is equal to $(x-1)(x-2) \dots (x-i+2) \left\{ x-i+1 - \frac{x}{i} \right\} = \frac{i-1}{i} \frac{(x-1)(x-2) \dots (x-i)}{x-i+1}$, and the identity is proved for this case.

Let the given series be represented by $f(x, i, j)$; then, obviously,

$$(x-2)(x-3) \dots (x-i) - \frac{x+1}{i-j} \cdot f(x, i, j) = f(x-1, i, j+1);$$

hence, assuming the truth for the value j ,

$$\begin{aligned} f(x-1, i, j+1) &= (x-2)(x-3) \dots (x-i) - \frac{j+1}{i} \cdot \frac{(x-1)(x-2) \dots (x-i)}{x+j-i} \\ &= (x-2)(x-3) \dots (x-i) \left\{ 1 - \frac{j+1}{i} \frac{x-1}{x+j-i} \right\} \\ &= \frac{i-j-1}{i} \frac{(x-2)(x-3) \dots (x-i-1)}{x+j-i}. \end{aligned}$$

Hence the identity is true for $j+1$, and therefore universally.

(2). *Otherwise*.—The right-hand expression E is of $i-1$ dimensions in x , and may therefore be expressed in terms of i other products as follows:

$$E = A_0(x-1)(x-2) \dots (x-i+1) + A_1x(x-1) \dots (x-i) + \dots + A_j(x+j-1)(x+j-2) \dots (x+j-i+1) + \dots + A_{i-1}(x+i-2) \dots x.$$

If in this identity we substitute for x the values 1, 2, 3, ... $(i-j-1)$, we see that all the coefficients after A_j vanish. By continuing this process we find the values of A_j, A_{j-1} , &c. But after the first two or three such coefficients the method becomes difficult, and merely serves to indicate the values of the remaining coefficients. To prove that the values thus indicated are true, put $x = i-j+p$ where p is a positive integer $< j$. Then it is necessary and sufficient to prove that for all such values of p

$$A_{j-p}(i-1)(i-2) \dots 1 + A_{j-p+1}i(i-1) \dots 2 + \dots \text{to } (p+1) \text{ terms} = 0, \\ \text{i.e., that } 1 - \frac{i}{1} \frac{p}{(i-p+1)} + \frac{i(i+1)}{1 \cdot 2} \frac{p(p-1)}{(i-p+1)(i-p+2)} + \dots = 0,$$

which is well known to be a fact. Thus the required theorem is entirely proved.

As the above theorem is one of considerable interest, I venture to add the following remarks:—

The series on which the solution is made to depend is a particular case of VANDERMONDE'S theorem, for which see Lond. Math. Society's *Proceedings*, Vol. IV, p. 286, and LIMBOURG'S *Théorie de la Fonction Gamma*, p. 124, in the latter of which works it is proved that

$$\begin{aligned} 1 - \frac{n}{1} \frac{\beta}{\alpha-n+1} + \frac{n(n-1)}{1 \cdot 2} \frac{\beta(\beta+1)}{(\alpha-n+1)(\alpha-n+2)} - \dots \\ = \frac{(\alpha-\beta-n+1)(\alpha-\beta-n+2) \dots (\alpha-\beta)}{(\alpha-n+1)(\alpha-n+2) \dots \alpha}, \end{aligned}$$

which, on putting $\beta = \alpha$, gives the required result.

The result may also be obtained very briefly as follows:—The series to be summed is a hyper-geometric series, and with the usual notation may be expressed as $F\{i, -p, i-p+1, 1\}$, and therefore by GAUSS' method of transformation it is $\frac{\Gamma(i-p+1)}{\Gamma(i+1)\Gamma(-p+1)}$, which vanishes since $-p+1$ is a negative integer.

If in this theorem we put $x = i+p$, where p is a positive integer, we get, by dividing down,

$$1 - \frac{j}{i-j+1} \frac{p+i}{p+1} + \frac{j(j-1)}{(i-j+1)(i-j+2)} \frac{(p+i)(p+i+1)}{(p+1)(p+2)} - \&c. = \frac{i-j}{i} \frac{p}{p+j}.$$

And if we put $x = -p$, where p is any positive integer, we get in like manner

$$1 - \frac{j}{i-j+1} \frac{p}{p+i-1} + \frac{j(j-1)}{(i-j+1)(i-j+2)} \frac{p(p-1)}{(p+i-1)(p+i-2)} - \&c. = \frac{i-j}{i} \frac{p+i}{p+i-j},$$

which two particular cases seem to be of interest.

6344. (By Professor WOLSTENHOLME, M.A.)—Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4 (1+2\cos^2 x)} F(\sin^2 x) dx = \frac{1}{2} \int_0^{\pi} F(\sin^2 x) dx \dots\dots\dots (1),$$

$$\int_{-\infty}^{\infty} \frac{\sin^6 x}{15-15\sin^2 x+2\sin^4 x} \frac{F(\sin^2 x)}{x^6} dx = \frac{1}{15} \int_0^{\pi} F(\sin^2 x) dx \dots\dots\dots (2).$$

Solution by Prof. NASH, M.A.; CHRISTINE LADD, B.A.; and others.

$$\log \sin x = \log x + \log \left(1 + \frac{x}{\pi}\right) + \log \left(1 - \frac{x}{\pi}\right) + \log \left(1 + \frac{x}{2\pi}\right) + \&c.$$

$$\text{Differentiating twice, } \cot x = \frac{1}{x} + \frac{1}{\pi+x} - \frac{1}{\pi-x} + \frac{1}{2\pi+x} - \frac{1}{2\pi-x} + \dots;$$

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \frac{1}{(\pi+x)^2} + \frac{1}{(\pi-x)^2} + \dots = \sum_{r=-\infty}^{\infty} \frac{1}{(x+r\pi)^2}$$

therefore, differentiating four times, we get

$$\frac{\cos x}{\sin^3 x} = 2 \frac{1}{(x+r\pi)^3}, \quad \frac{1+2\cos^2 x}{\sin^4 x} = 3 \sum \frac{1}{(x+r\pi)^4} \dots\dots\dots (A),$$

$$\frac{\cos x (2+\cos^2 x)}{\sin^5 x} = 3 \sum \frac{1}{(x+r\pi)^5}, \quad \frac{15-15\sin^2 x+2\sin^4 x}{\sin^6 x} = 15 \sum \frac{1}{(x+r\pi)^6} \dots\dots\dots (B).$$

The first integral is evidently equal to

$$\int_0^{\pi} \frac{\sin^4 x}{1+2\cos^2 x} F(\sin^2 x) dx \left\{ \frac{1}{x^4} + \frac{1}{(\pi+x)^4} + \frac{1}{(\pi-x)^4} + \frac{1}{(2\pi+x)^4} + \frac{1}{(2\pi-x)^4} + \dots \right\} \\ = \frac{1}{2} \int_0^{\pi} F(\sin^2 x) dx, \text{ by equation (A).}$$

Similarly (2) follows from equation (B).

6263. (By the Rev. H. G. DAY, M.A.)—If a line of length c is divided into n parts, $x, y, z, \dots$, by $(n-1)$ points taken at random in it, prove that the average values (1) of $xyz \dots$, (2) of $x^\alpha y^\beta z^\gamma \dots$, are

$$\frac{(n-1)!}{(2n-1)!} c^n, \quad \frac{(n-1)! \alpha! \beta! \gamma! \dots}{(n-1+\alpha+\beta+\gamma \dots)!} c^{\alpha+\beta+\gamma+\dots}$$

Solution by Prof. NASH, M.A. ; Prof. MATZ, M.A. ; and others.

Taking $x_1, x_2 \dots x_{n-1}$ as distances of points of section from one end, and writing $a_1, a_2 \dots a_n$ for $\alpha, \beta, \gamma \dots$, the probability in (2) is

$$\int_0^c \int_{x_1}^c \dots \int_{x_{n-1}}^c x_1^{a_1} (x_2 - x_1)^{a_2} (x_3 - x_2)^{a_3} \dots (x_{n-1} - x_{n-2})^{a_{n-1}} (c - x_{n-1})^{a_n} \\ \times dx_1 dx_2 \dots dx_{n-1} + \int_0^c \int_{x_1}^c \dots \int_{x_{n-1}}^c dx_1 dx_2 \dots dx_{n-1}.$$

$$\text{Now } \int_a^c (x-a)^m (c-x)^n dx = \frac{n! m!}{(n+m+1)!} (c-a)^{m+n+1};$$

$$\therefore \text{first integral} = \int_0^c \int_{x_1}^c \dots \int_{x_{n-3}}^c x_1^{a_1} (x_2 - x_1)^{a_2} \dots (c - x_{n-2})^{a_n + a_{n-1} + 1} \\ \times dx_1 dx_2 \dots dx_{n-2} \times \frac{a_n! a_{n-1}!}{(a_n + a_{n-1} + 1)!} \\ = \frac{a_n! a_{n-1}!}{(a_n + a_{n-1} + 1)!} \cdot \frac{(a_n + a_{n-1} + 1)! a_{n-2}!}{(a_n + a_{n-1} + a_{n-2} + 2)!} \int_0^c \int_{x_1}^c \dots \int_{x_{n-4}}^c x_1^{a_1} (x_2 - x_1)^{a_2} \dots \\ \dots (c - x_{n-3})^{a_n + a_{n-1} + a_{n-2} + \dots} dx_1 dx_2 \dots dx_{n-3} = \&c. \\ = \frac{a_n! a_{n-1}! a_{n-2}! \dots a_1!}{(a_n + a_{n-1} + a_{n-2} + \dots + a_1 + n - 1)!} c^{a_n + a_{n-1} + \dots + a_1 + n - 1}.$$

$$\text{In second integral, } a_1 = a_2 = \dots = 0: \therefore \text{integral} = \frac{1}{(n-1)!} c^{n-1}$$

$$\therefore \text{probability} = \frac{a_n! a_{n-1}! \dots a_1! (n-1)!}{(a_n + a_{n-1} + \dots + a_1 + n - 1)!} \cdot c^{a_n + a_{n-1} + \dots + a_1}.$$

$$\text{In case (1), } a_1 = a_2 = \dots = a_n = 1, \therefore \text{probability} = \frac{(n-1)!}{(2n-1)!} c^n.$$

[Another solution is given on p. 52 of Vol. 36 of the *Reprints*.]

7002. (By Professor SYLVESTER, F.R.S.)—Let s be any series of positive integers beginning with 0 and ending with n , subject to the condition that no term shall exceed the mean between the term which precedes and the term which follows it; show that the number of such series s is equal to the number of indefinite partitions of n .

Solution by the PROPOSER ; W. J. C. SHARP, M.A. ; and others.

Let $a, b, c, d \dots l$ be any partition of n arranged in ascending order; then $0, a, a+b, a+b+c, a+b+c+d \dots +l$ will be one of the series s .

For, by construction, b is not less than a , and therefore a is not greater than $\frac{1}{2}(a+b)$; c is not less than b , and therefore $(a+b)$ is not greater than $\frac{1}{2}(2a+b+c)$, and so on; also $a+b+c \dots + i = n$ by construction.

Again, let $0, a, \beta, \gamma \dots, \lambda, \mu, \nu$ be any series s ; then $a, \beta-a, \gamma-\beta, \delta-\gamma, \mu-\lambda, \nu-\mu$ will be a partition of n written in ascending order. For

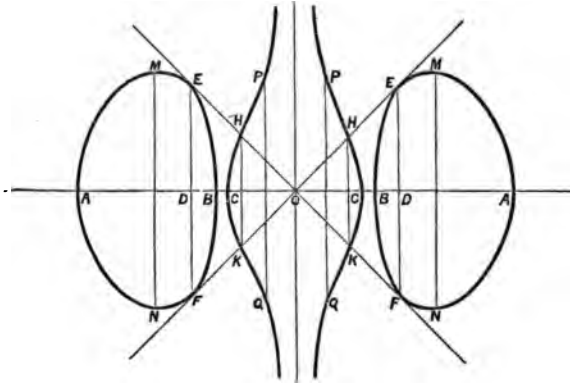
$$\beta-a = \text{or} > a, \gamma-\beta = \text{or} > \beta-a, \dots \gamma-\mu = \text{or} > \mu-\lambda.$$

Hence, collecting these two statements, to every partition of n written in ascending order corresponds one s , and to every s corresponds one partition of n written in ascending order.

Hence the number of series s equals the number of partitions of n written in ascending order, or, which comes to the same thing, is equal to the number of partitions of n .

ON THE ATRIPHTHALOID AND ATRIPHTHALID OF DR. HAUGHTON.
(Quests. 6977, 7003, 7052.) By Professor TOWNSEND, F.R.S.

Dr. HAUGHTON, having obtained on dynamical principles a class of curves as the possible meridians of the free surface of a frictionless ocean in steady oscillatory motion about its permanent form of stable equilibrium on the attracting mass of a revolving sphere, has called them in consequence Atripththalassic curves. The simplest of the class is that having for equation $r^2(A-2gr) = B \operatorname{cosec}^2 \theta$, where r and θ are the radius and polar distance of any point on the curve, A and B two independent positive constants, and g the superficial gravitation of the sphere. This he has called in particular the Atripththalid; and the discussion of its properties is the object of the present article.



Discussion of the Curve $r^2(A-2gr) = B \operatorname{cosec}^2 \theta$.

Putting, for convenience of discussion, $A = 2gh$, $B = 2gk^2$, and $\theta = 90^\circ - \omega$, the equation of the curve assumes the form

$$r^2(r-h) + k^2 \sec^2 \omega = 0 \dots\dots\dots(1),$$

where h and k are positive constants representing linear magnitudes,

which may have any independent values from 0 to ∞ , and which may be regarded as the parameters of the curve.

Since (1) gives the same values for r when ω is changed into $-\omega$, or into $\pi \pm \omega$, we see at once the curve it represents is symmetrical with respect to the two rectangular axes $\omega = 0$ and $\omega = \frac{1}{2}\pi$, and has a centre at the origin O. [See fig. for this and subsequent references.]

The equation (1) being, for every value of ω , a cubic in r whose absolute term is positive: hence, for every value of ω , r has one real negative value, commencing from its minimum absolute value OC when $\omega = 0$, and increasing continuously to ∞ from $\omega = 0$ to $\omega = \frac{1}{2}\pi$. Hence the two symmetrical conchoidal-shaped infinite branches, which always meet asymptotically on the axis for which $\omega = \frac{1}{2}\pi$, and which never disappear for any finite values of h and k however related to each other.

The remaining two roots of the cubic (1) for r in terms of ω being real or imaginary, by the theory of equations, according as $\sec^2 \omega$ is

$< \text{or} > \frac{4}{27} \frac{h^3}{k^3}$; hence, when $\frac{4}{27} \frac{h^3}{k^3}$ is > 1 , the curve, in addition to the

two infinite symmetrical conchoidal branches which never disappear, has two finite symmetrical ovals, lying entirely outside the conchoidal branches, and intersecting the axis of r at two pairs of real points A and B. The two tangents OE and OF to which from the origin O (see fig.) are given

in position by the equation $\sec^2 \omega = \frac{4}{27} \frac{h^3}{k^3}$, and in magnitude by the

equation $O = OF = E\frac{1}{3}h$, and the equal distances OD of their chords of contact EF from the origin by the equation $OD^2 = 3 \frac{h^3}{k^3}$.

When $\frac{4}{27} \frac{h^3}{k^3} = 1$, then for the two tangents $\sec^2 \omega = 1$, and the ovals consequently contract into points, whose equal distances from the origin $= \frac{1}{3}h$: the pairs of vertices A and B of the two ovals then coincide, and the equal distances OC of the two conchoidal vertices from origin $= \frac{1}{3}h$.

When $\frac{4}{27} \frac{h^3}{k^3}$ is < 1 , then for the two tangents $\sec^2 \omega < 1$, and therefore ω is imaginary. The two ovals then disappear altogether, and the curve consists entirely of the two conchoidal branches, which never disappear.

The sum of the three roots r_1, r_2, r_3 of (1) being independent of the value of ω , and $= h$ for all directions of r ; hence, for the two tangents, since $r_1 = r_2 = \frac{2}{3}h$ for the ovals, therefore $r_3 = -\frac{1}{3}h$ for the conchoidal branches. Hence (see fig.) the tangential radii OE and OF of the ovals, when real, are bisected at H and K by the conchoidal branches at their sides of the asymptotic axis of the curve.

The sum of the three products in pairs $r_2 r_3 + r_3 r_1 + r_1 r_2$ of the three roots r_1, r_2, r_3 of (1) being also independent of ω , and $= 0$ for all directions of r ; therefore, for every three radii of the curve having a common direction, if r_1 and r_2 be those to either oval, and r_3 that to the infinite branch at the same side of the asymptotic axis, $r_3(r_1 + r_2) = r_1 r_2$. Hence, for every three radii having a common direction, that to either infinite branch is in magnitude and direction half the harmonic mean between those to the oval at the same side of the asymptotic axis of the curve.

Differentiating (1) with respect to ω , we get

$$\frac{dr}{d\omega} = -2k^3 \cdot \frac{\sin \omega}{\cos^3 \omega} \cdot \frac{1}{3r-2h} \cdot \frac{1}{r^2} \dots\dots\dots(2),$$

which shows that $\frac{dr}{r d\omega} = 0$ only when $\sin \omega = 0$; that is, only for the three apsidal points A, B, C of the curve. Hence the radius r of the curve has its maxima and minima values only at the three apsidal points on the axis for which $\omega = 0$, and the conchoidal branches have in consequence no dumb-bell depressions throughout their whole lengths.

Since, from (2), $\frac{dr}{r d\omega} = \infty$ when $\omega = \frac{1}{2}\pi$ and when $3r - 2h = 0$; therefore, as already observed, the axis for which $\omega = \frac{1}{2}\pi$ is an asymptotic tangent to the infinite branches, and the two radii OE and OF for which $3r = 2h$ are the two ordinary tangents to the ovals from the origin.

Putting, in (1), $x = r \cos \omega$ and $y = r \sin \omega$, that is, transforming into rectangular coordinates, and solving for y , we get

$$y^2 = h^2 - x^2 - 2hk^2x^{-2} + k^6x^{-4} \dots\dots\dots(3),$$

which shows that, for every finite value of x^2 , y^2 has but a single finite value, positive or negative; that, from $x = \infty$ to $x = OA$, y^2 is negative, and therefore y imaginary; that, from $x = OA$ to $x = OB$, y^2 is positive and finite, and therefore y real and finite; that, from $x = OB$ to $x = OC$, y^2 is negative, and therefore y imaginary; and that, from $x = OC$ to $x = 0$, y^2 is positive and increasing from 0 to ∞ , and therefore y real and increasing from 0 to ∞ . These results obviously verify the form of the curve as obtained above from the equation (1) for r .

Putting $y = 0$ in (3), we obtain for the squares of the three semi-axes a , b , c of the curve, that is of the three distances OA, OB, OC of the three vertices A, B, C from the origin O, the cubic equation

$$x^6 - h^2x^4 + 2hk^2x^2 - k^6 = 0 \dots\dots\dots(4),$$

which has always one real positive root, and whose three roots when real are all positive.

The equation (4) is, by the theory of equations, that whose roots are the squares of those of (1) for the case when $\omega = 0$.

Since, from (1), when $\omega = 0$, by the theory of equations,

$$a + b + c = h, \quad bc + ca + ab = 0, \quad abc = -k^3,$$

$$\text{therefore} \quad c = -\frac{ab}{a+b}, \quad h = \frac{a^2 + ab + b^2}{a+b}, \quad k^3 = \frac{a^2b^2}{a+b} \dots\dots\dots(5),$$

and, if d denote OD, since $d^2 = 3 \frac{k^2}{h}$, therefore $d^2 = 3 \frac{a^2b^2}{a^2 + ab + b^2}$; rela-

tions which give the values of the four quantities c and d , h and k , in terms of a and b , and show consequently that the two latter quantities, which when real may have any independent values from 0 to ∞ , determine completely all the particulars of the curve in every case.

In the particular case when $4h^3 = 27k^3$, that is when the two ovals contract into points, and when consequently $a = b$, equation (4) has two of its roots each $= \frac{2}{3}h^2$, and its third root $= \frac{1}{3}h^2$.

In the particular case when $h = 2k$, in which as $4h^3 > 27k^3$ the two ovals are real, the three roots of (4) are respectively $a^2 = k^2$, and $a^2 = \frac{1}{2}(3 \pm \sqrt{5})k^2$.

In the same case, the roots of (1), when $\omega = 0$, are respectively $r = k$, and $r = \frac{1}{2}(1 \pm \sqrt{5})k$; as they ought, the roots of (4) being in every case the squares of those of (1), when, in the latter, $\omega = 0$.

Putting in for h and k in the function $(4h^3 - 27k^3)k^3$, on whose sign as positive or negative it depends whether the ovals are real or imaginary,

their values in terms of a and b as given by (5), we find readily that

$$(4h^3 - 27k^3)k^3 = \left[\frac{[2(a^2 + ab + b^2) + 3ab]ab}{(a+b)^2} \right]^2 (a-b)^3 \dots\dots\dots (6),$$

and that its sign is consequently, as it ought to be, positive or negative according as a and b are real or imaginary.

The function in question being, by the theory of equations, the product of the squares of the differences of the roots of (1) for the case when $\omega = 0$, the same value for it in terms of a and b would be obtained even more readily than above by the substitution for c in that product of its value in terms of a and b . See equations (1) and (6).

Multiplying both sides of (3) by πdx , integrating between the limits a and b , and substituting in the result for h and k their values just given in terms of a and b ; we get, for the volume V of the solid generated by the revolution of either oval round its axis of figure AB, the value (7) below in terms of a and b , which, compared with that of the volume S of the sphere on AB as diameter, viz., $\frac{4}{3}\pi(a-b)^3$, gives for the ratio of the two volumes in terms of a and b the subjoined value (8):—

$$V = \frac{4}{3}\pi \left(\frac{a-b}{a+b} \right)^2 (a^3 - b^3), \quad \frac{V}{S} = \frac{4}{3} \frac{a^2 + ab + b^2}{(a+b)^2} \dots\dots\dots (7, 8).$$

This value (8), lying always between the extreme limits 3 and 4 corresponding respectively to the extreme values 1 and ∞ of the unrestricted ratio of a to b , shows, consequently, that the extreme depth MN is always greater than the extreme breadth AB of the ovals. That the chords of contact EF of the tangents to the ovals from the centre O of the curve which, when the ovals are finite, are of course always less than their extreme depths MN, are also in all cases greater than their extreme breadths AB, may be readily shown from (6) as follows. Since

$$EF^2 = 4(OE^2 - OD^2) = 4(OF^2 - OD^2) = 4\left(\frac{4}{3}h^3 - 3\frac{k^3}{h}\right) = \frac{4}{3}\left(\frac{4h^3 - 27k^3}{h}\right),$$

therefore from (6), as $(a-b) = AB$, and as $hk^3 = a^2b^3(a^2 + ab + b^2) + (a+b)^2$

$$\text{by (5),} \quad \left(\frac{EF}{AB} \right)^2 = \frac{4}{3} \frac{[2(a^2 + ab + b^2) + 3ab]^2}{(a^2 + ab + b^2)(a+b)^2} \dots\dots\dots (9),$$

a ratio always exceeding unity, and having the extreme values 27 + 9 and 16 + 9 for the extreme values 1 and ∞ of the unrestricted ratio of a to b .

The value of MN in terms of a and b not being in general determinable like that of EF in finite terms, the exact value of the ratio of MN to AB cannot consequently be given in general in such terms. As EF is never greater than MN when both are real, and only equal to it when both are evanescent, the least possible value of the ratio of EF to AB given by (9), viz., 4 + 3, is consequently an inferior though by no means a close limit to the possible ratio of MN : AB. But a superior as well as a closer inferior limit, both however much outside the extreme limits of the actual ratio, may be found for it, from equation (7) as follows.

As the volume V of the solid generated by the revolution of either oval round its axis of figure AB must necessarily be less than that of the circumscribed cylinder, and greater than that of the inscribed double cone having AB for axis and the circle described by MN for base, we must therefore have, for all values of a and b , as $(a-b) = AB$

$$\frac{4}{3}\pi \left(\frac{a-b}{a+b} \right)^2 (a^3 - b^3) < \frac{1}{4}\pi \cdot MN^2 \cdot AB > \frac{1}{12}\pi \cdot MN^2 \cdot AB,$$

and therefore $\left(\frac{MN}{AB}\right)^2 < 8 \frac{a^2 + ab + b^2}{(a+b)^2} > \frac{1}{8} \frac{a^2 + ab + b^2}{(a+b)^2}, \dots\dots\dots(10);$

from which it follows, consequently, that the square of the ratio in question must always lie between the extreme limits 2 and 8, its least and greatest values corresponding respectively to the least and greatest values 1 and ∞ of the unrestricted ratio of a to b . From the manner in which they have been obtained, however, these limits are obviously much outside of those of the actual ratio.

Differentiating (3) with respect to x ,

$$y \frac{dy}{dx} = -x + 2hk^2x^3 - 2k^4x^5, \dots\dots\dots(11);$$

from which we get, for the values of x^2 for which $y \frac{dy}{dx} = 0$ and for which consequently y^2 is a maximum or a minimum, the cubic equation

$$x^6 - 2hk^2x^2 + 2k^4 = 0, \dots\dots\dots(12),$$

which has always one real negative root for which x is consequently imaginary, and whose remaining roots will, by the theory of equations, be real or imaginary according as $8h^3 - 27k^4$ is $>$ or $<$ 0, and of the same sign when real. Hence the curve has never more than two values of x^2 for which y^2 is a maximum or a minimum; and as those values of x^2 lie necessarily, one between the limits a^2 and b^2 for which y^2 is positive, and the other between the limits b^2 and c^2 for which y^2 is negative, there is therefore never more than one pair of maxima double ordinates to the curve which are real as regards both position and magnitude, viz., those MN of the ovals when real.

In the particular case when $4h^3 = 27k^4$, that is when the two ovals contract into points, the three roots of (6) are, respectively,

$$x^2 = \frac{4}{3}h^2, \quad x^2 = \frac{2}{3}(\sqrt{3}-1)h^2, \quad \text{and} \quad x^2 = -\frac{2}{3}(\sqrt{3}+1)h^2,$$

the first only of which gives a pair of chords real as regards both position and magnitude of the curve.

Eliminating x^2 between (3) and (12), we get, after a few reductions, for the actual maxima and minima values of y^2 , the cubic equation

$$4y^6 - 8h^2y^4 + 4h(h^3 - 9k^3)y^2 + k^3(4h^3 - 27k^4) = 0 \dots\dots\dots(13),$$

which has always one real positive root corresponding to the real negative root of (12), and belonging consequently to no chords real even as regards position of the curve; and whose remaining roots will be real and have opposite signs when $4h^3 - 27k^4$ is positive, that is when the two ovals are real and finite.

In the particular case when $4h^3 = 27k^4$, that is when the ovals contract into points, one root of (13) is evanescent, and corresponds to the evanescent ovals; and the remaining two are respectively $= h^2(1 \pm \frac{2}{3}\sqrt{3})$; which correspond respectively, the former to the real negative root of (12) and therefore to no real chords even as regards position of the curve, and the latter to the pair of chords between the evanescent ovals and the infinite branches, which, though real as regards their positions, have no real intersections with the curve.

Differentiating with respect to x the value of $\frac{dy}{dx}$ given by (11), having first substituted in it for y its value in x from (3), and equating the result to 0, we get, after a few ordinary reductions, for the values of x^2 for which $\frac{d^2y}{dx^2} = 0$, that is for the several points of inflexion, real or

imaginary, of the curve, the equation of the sixth degree

$$x^6 [k^2 x^3 - 12hk^2 x^2 + 3k^3 (2h^3 + 6k^3) x^2 - 18h^2 k^2 x^4 + 18hk^3 x^2 - 6k^{12}] = 0, \dots (14)$$

of whose roots one is obviously evanescent, one essentially real and positive, and the remaining four when real essentially positive also.

The evanescent root corresponds of course to the axis of y , which is consequently a doubly inflexional as well as asymptotically tangential chord of the curve, and the essentially real and positive root to the pair of inflexional chords PQ of the conchoidal branches; which, equidistant in opposite directions from the axis of y , lie necessarily somewhere between it and the parallel tangents at the vertices C of the branches. The four remaining roots, with the four pairs of inflexional chords reflexions of each other with respect to the axis of y to which they correspond in the curve, may be all real, or all imaginary, or two real and two imaginary, according to the particular value of the parametric ratio $h + k$ on which alone they depend. When the two ovals are real and finite, as they are for all values of the ratio for which $4h^3 > 27k^3$, the entire number of pairs of inflexional chords intersecting them at pairs of real points is necessarily even; but none of the four pairs, even when themselves real as regards their positions, need intersect them necessarily at real points at all. So that the ovals may be, and in fact often if not always are, as represented in the figure, concave to their interiors throughout the entire circuits of their perimeters.

The application of Sturm's theorem to (14) gives us, for all numerical values of the parametric ratio, the exact numbers of corresponding pairs of inflexional chords which occupy real positions within any two assigned limits of distance from the centre O of the curve; and, as the corresponding values of OA and OB can also be determined for all such values to any degree of approximation, from equation (1) or (4), by Horner's and other methods of numerical solution, the exact numbers of real pairs of inflexional chords lying within the intervals AB, and therefore intersecting the ovals at real points, can consequently be determined by its aid for all numerical values of the ratio. Its applications, however, are in general laborious, and in the present instance uninteresting except for such values of the ratio.

By its application to the equation for the two particular cases when $h + k = 1$ and 2 respectively, which correspond, the former to an imaginary and the latter to a real pair of ovals; we find, with comparatively little trouble in either case, arising from the circumstance of the quadratic functions having imaginary roots in both, and therefore dispensing with the necessity of proceeding any further with the process in either, that the equation, in addition to its evanescent, has for each of them but a single real root, that viz., corresponding to the pair of inflexional chords of the infinite branches which are always real. And, by a similar application, attended with a little more trouble arising from the reality of the roots of the quadratic functions in each case, we arrive at the same result for the two cases when $4h^3 = 26k^3$ and $28k^3$ respectively, which correspond again, the former to an imaginary and the latter to a real pair of ovals; the intermediate case for which $4h^3 = 27k^3$ being that for which they pass through evanescence from their real to their imaginary state, and conversely.

In the particular case when $4h^3 = 27k^3$, for which the ovals contract into points, the five finite roots of (14) are, on the contrary, all real, and correspond in consequence to five pairs of chords all real as regards their positions; one of which intersects, as in all cases, the conchoidal branches, and, of the remaining four, three coincide at the evanescent ovals, and

the fourth intersects the curve at imaginary points. For the equation is easily seen to be then equivalent to

$$\lambda^2 x^2 (x^2 - \frac{1}{3}\lambda^2)^3 [x^4 - \frac{1}{3}\lambda^2 x^2 + \frac{1}{3}(\frac{1}{3}\lambda^2)^2] = 0, \dots\dots\dots(15),$$

which, besides its single root $= 0$, has evidently three roots each $\frac{1}{3}\lambda^2$, and two others equal respectively to $\frac{1}{3}\lambda^2(1 \pm \sqrt{\frac{1}{3}})$, which correspond, respectively, the three former to the evanescent ovals, the latter with the lower sign to the conchoidal branches, and the latter with the upper sign to no real points at all on the curve.

Of the three pairs of chords coinciding at the evanescent ovals in this case, two however correspond to the evanescent radii of curvature at their vertices; the function $-y^3 \frac{d^2 y}{dx^2}$, which multiplied by x^{10} is mani-

festly equivalent to the quantity within the brackets at the left side of equation (14), representing in all cases, as is well known, at the several vertices of any curve symmetrical with respect to the axis of x , the squares of the corresponding radii of curvature at the vertices, and being consequently evanescent at the two coincident vertices of every acnodal double point on the axis of the curve. As regards the third pair in the same case; taking it into connexion with the pair intersecting the curve at imaginary points, and conceiving both pairs to change position together with the gradual and continued increase of the parametric ratio from its critical to every higher value, and the consequent accompanying dilatation of the ovals from their evanescent to every greater magnitude; they are to be regarded, while real, as two variable pairs of inflexional chords intersecting the expanding ovals at pairs of imaginary points, and after coming together in the course of their variation, as the above particular cases show they do very rapidly with the increase of the ratio, then passing through coincidence from their real to their imaginary state, beyond which the particulars above stated supply no clue to follow them. The questions, as to whether for any values of the parametric ratio the ovals have ever real points of inflexion, and as to the critical values (if any) of the ratio for which they cease (if ever) to be, as represented in the figure, concave to their interiors throughout the entire circuits of their perimeters, have, it will be observed, been left undecided in the above investigation.

7029. (By Professor MALET, M.A.)—If through a focus of an ellipsoid of revolution any three mutually rectangular chords be drawn, prove that the sum of their reciprocals is constant.

Solution by Dr. CURTIS; W. H. BLYTHE, M.A.; and others.

The length of a focal chord of any conic is proportional to the rectangle under the segments into which the chord is divided by the focus, or, again, to the square of the parallel semidiameter; hence, as in any ellipsoid, or hyperboloid, the sum of the reciprocals of three semi-diameters mutually at right angles is constant, the theorem results.

7049. (By T. WOODCOCK, B.A.)—Prove that

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \dots \text{ad. inf.} = \frac{1}{2\sqrt{2}} \left\{ \frac{1}{2}\pi + \log(\sqrt{2}-1) \right\}.$$

Solution by R. RAWSON; R. KNOWLES, B.A., L.C.P.; and others.

$$\text{The series is } \frac{1}{3} - \frac{1}{3+4} + \frac{1}{3+2 \cdot 4} - \frac{1}{3+3 \cdot 4} + \dots \text{ad. inf.} \dots\dots\dots(1),$$

$$\text{a particular case of } \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots \text{ad. inf.} \dots\dots\dots(2),$$

where a, b are any numbers whatever.

Using LORONA's method as given in YOUNG's *Int. Calc.*, pp. 99 to 107, the sum is given by (3) below, the series coinciding with (2) when $x = 1$,

$$S = \frac{x^a}{3} - \frac{x^{a+b}}{a+b} + \frac{x^{a+2b}}{a+2b} - \frac{x^{a+3b}}{a+3b} + \dots \text{ad. inf.} \dots\dots\dots(3),$$

$$\frac{dS}{dx} = x^{a-1} [1 - x^b + x^{2b} - x^{3b} + \dots \text{ad. inf.}] = \frac{x^{a-1}}{x^b + 1}, \therefore S = \int_0^1 \frac{x^{a-1} dx}{x^b + 1} \dots\dots\dots(4).$$

The value of S in (4) is evidently the sum of the series (2). The integral (4) is readily obtained, generally, by means of DE MOIRVE's theorem, and is given by all writers on the Integral Calculus. If $a=3, b=4$, (4) becomes

$$S = \int_0^1 \frac{x^2 dx}{x^4 + 1} = \frac{1}{2\sqrt{2}} \log \left(\frac{2-\sqrt{2}}{2+\sqrt{2}} \right) + \frac{2}{4\sqrt{2}} \cdot \frac{1}{2}\pi,$$

which agrees with the result in the question.

[Several interesting results follow by giving various values to a and b .]

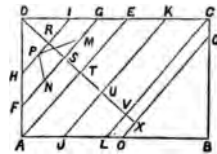
6135. (By W. S. B. WOOLHOUSE, F.R.A.S.)—Find (1) the average area of triangles drawn on the surface of a given rectangle, and having one of the three sides parallel to a given line; and (2) the directions of the given line when the average area is a maximum and a minimum.

Solution by Prof. SEITZ, M.A.; Prof. MATZ, M.A.; and others.

1. Let ABCD be the given rectangle, AE, FG, HI, JK, LC, OQ lines drawn parallel to the given line, and DRSTUVX a line perpendicular to these lines. Form a triangle by joining any two points in FG, as M, N, with any point P in HI.

Let $AB=a, AD=b, DT=h, DV=k, DS=w, DR=x, FM=y, MN=z, FG=y_1, AE=JK=y_2$, and $\angle DAE = \phi$.

Then $h = b \sin \phi, k = a \cos \phi, y_1 = w \operatorname{cosec} \phi \sec \phi$, the area of MNP is $= \frac{1}{2}(w-x)z$, if $x < w$, and $= \frac{1}{2}(x-w)z$, if $x > w$. For any value of x



7042. (By J. GRIFFITHS, M.A.)—If m and n are positive integers ($m > n$), show that

$$\frac{2^{-n}}{n!} + \frac{(m-n)(m-n-1)}{(n+1)!} \cdot 2^{-(n+2)} + \frac{1}{2!} \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{(n+2)!} \cdot 2^{-(n+4)} + \dots$$

$$= \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2m-1)}{(m+1)!}.$$

Solution by the Rev. T. R. TERRY, M.A.; SARAH MARKS; and others.

Using the known properties of the hypergeometric series, we have

$$\text{Left-hand side} = \frac{1}{2^n \cdot n!} F\left[\frac{1}{2}(n-m), \frac{1}{2}(n+1-m), n+1, 1\right]$$

$$= \frac{1}{2^n \cdot n!} \frac{\Gamma(n+1) \Gamma[\frac{1}{2}(2m+1)]}{\Gamma[\frac{1}{2}(m+n+2)] \Gamma[\frac{1}{2}(m+n+1)]}.$$

But $\Gamma(n+1) = n!$, $\Gamma[\frac{1}{2}(2m+1)] = 2^{-m} (2m-1)(2m-3) \dots 1 \Gamma(\frac{1}{2})$,

$$\Gamma[\frac{1}{2}(m+n+2)] \Gamma[\frac{1}{2}(m+n+1)] = 2^{-(m+n)} (m+n)! \Gamma(\frac{1}{2}),$$

whence the required result.

7028. (By Professor TOWNSEND, F.R.S.)—Assuming Dr. C. TAYLOR's theorem, that, for a curve of class p , the locus of the intersections of orthogonal tangents is of degree $p(p-1)$; show that, for two curves of classes p and q , the locus of the intersections of tangents to either with orthogonal tangents to the other is of degree $2pq$.

Solution by Prof. GENESE, M.A.; CHRISTINE LADD, B.A.; and others.

The two curves may be considered as forming together an envelope of class $(p+q)$. Therefore any straight line will cut the locus of intersection of tangents at right angles in $(p+q)(p+q-1)$ points. But of these $p(p-1)$ are tangents to one curve only, and $q(q-1)$ to the other, &c.

The theorem assumed may be demonstrated as follows:—

Let $y = m(x-a)$ be the equation to a tangent from the point $(0, a)$ to a curve of class p . Then m is a root of $a_p m^p + a_{p-1} m^{p-1} + \dots + a_0 = 0$, where any coefficient, say a_r , is a function of a of degree r at most. Let $R = 0$ be the condition that the equation in m should have two roots whose product is -1 . The degree of R in a gives the number of points in which the locus of the intersection of tangents at right angles meets $x = 0$.

Now the problem in analysis is the same, if for the curve of class p we substitute a system of p points. Each pair of points gives two points of the locus on $x = 0$. The degree of the locus is therefore

$$\frac{1}{2} p(p-1) \times 2 = p(p-1).$$

The result in Quest. 7028 is obtained by replacing the two curves by systems of p and q points.

It may similarly be shown that the locus of the intersection of tangents including any constant angle (except the right) is of degree $2p(p-1)$, and the corresponding locus for two curves is of degree $4pq$.

Again, Dr. TAYLOR's theorem shows that $R = 0$ is of degree $p-1$ in the coefficients $a_0, a_1, \&c.$ (This may be easily verified for $p=3, 4$, and 5 .)

The theorem may therefore be stated thus:—

If p be the class of a curve, q the number of tangents to it parallel to any given direction (*i.e.*, the degree in a of $a_p m^p + a_{p-1} m^{p-1} + \&c.$), then the degree of the locus of the intersection of tangents at right angles is $q(p-1)$.

The locus can meet the line at infinity only at the imaginary circular points or focoids.

[Prof. GENÈSE remarks that the analytical investigation is interesting as leading to properties of a new class of determinants which may be called orthogonants or orthocyclants.]

7046. (By S. CONSTABLE.)—Prove that the three straight lines each passing through the centre of an escribed circle of a triangle, and through the middle point of the corresponding base, pass through a common point. (Suggested by the EDITOR's Question 6984.)

Solution by S. N. WILLIS; W. H. BLYTHE, M.A.; and others.

If D, E, F be the mid-points of the sides of the triangle ABC , the lines O_1D, O_2E, O_3F are seen immediately to be parallel to the lines AH, BJ, CL , where H, J, L are the points of contact of the inscribed circle (see Fig. to Quest. 6984); but this latter triad of lines meet in a point, therefore the former triad, viz., O_1D, O_2E, O_3F , meet in a point.

[The theorem is also true of a spherical triangle, the lines being arcs of great circles.]

6997. (By J. L. MCKENZIE, B.A.)—If Δ be the area of the triangle ABC , prove that

$$4\Delta^2 = abc(a+b+c)(1 - \sin^2 \frac{1}{2}A - \sin^2 \frac{1}{2}B - \sin^2 \frac{1}{2}C);$$

$$\text{and } 8\Delta^3 = \frac{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2) \sin^2(A+B) \sin^2(B+C) \sin^2(C+A)}{\sin(A-B) \sin(B-C) \sin(C-A)}.$$

Solution by the Rev. T. R. TERRY, M.A.; Dr. CURTIS; and others.

Putting E, F for the right-hand expressions, we have, since

$$2 \sin A = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C, \text{ and } 1 - 2 \sin^2 \frac{1}{2}A = 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C,$$

$$E = 16R^2 \sin^2 A \sin^2 B \sin^2 C = 4\Delta^2.$$

Again, since $\frac{a^2 - b^2}{\sin(A-B)} = 4R^2 \cdot \frac{\sin^2 A - \sin^2 B}{\sin(A-B)} = 4R^2 \sin C$,
therefore $F = 64R^2 \sin^2 A \sin^2 B \sin^2 C = 8\Delta^2$.

6969. (By R. TUCKER, M.A.)—AP, AQ are two vertical chords of a parabola, which are complementally inclined to the axis, and ϕ, ϕ' are the angles made by the tangents at P, Q with AP, AQ; prove that
 $\cot \phi \sim \cot \phi' = 2\Delta APQ + (\text{Latus Rectum})^2$.

Solution by J. S. JENKINS; R. KNOWLES, B.A.; and others.

Let $(x_1, y_1), (x_2, y_2)$ be the coordinates of P, Q respectively; then $\angle TPA = \phi$, $\angle TQA = \phi'$, and $\angle QAN = \frac{1}{2}\pi - \angle PAM$; hence

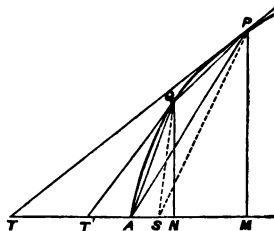
$$\tan \angle QAN = \cot \angle PAM, \therefore (x_1 x_2)^{\frac{1}{2}} = 4a.$$

$$\begin{aligned} \text{Now } 2 \text{ area } \triangle APQ &= x_1 y_2 - x_2 y_1 \\ &= 2 (ax_1 x_2)^{\frac{1}{2}} (x_1^{\frac{1}{2}} - x_2^{\frac{1}{2}}) \\ &= 8a a^{\frac{1}{2}} (x_1^{\frac{1}{2}} - x_2^{\frac{1}{2}}) = 16a^2 \left(\frac{x_1^{\frac{1}{2}} - x_2^{\frac{1}{2}}}{2a^{\frac{1}{2}}} \right). \end{aligned}$$

$$\text{Again, } \cot \phi = \frac{2x_1 + 4a}{y_1}, \quad \cot \phi' = \frac{2x_2 + 4a}{y_2},$$

$$\text{therefore } \cot \phi - \cot \phi' = \frac{4a a^{\frac{1}{2}}}{8a^2} (x_1^{\frac{1}{2}} - x_2^{\frac{1}{2}}) = \frac{x_1^{\frac{1}{2}} - x_2^{\frac{1}{2}}}{2a^{\frac{1}{2}}}.$$

$$\text{Hence } 2 \text{ area } \triangle APQ + (4a)^2 = \cot \phi - \cot \phi'.$$



6934. (By G. F. WALKER, M.A.)—Solve the first set, and eliminate x, y, z from the second set, of the following equations:—

$$\begin{aligned} (x+x-y)(x+y-z) &= ax, & (x+x-y)(x+y-z) &= ayz, \\ (x+y-z)(y+z-x) &= by, & (x+y-z)(y+z-x) &= bzx, \\ (y+z-x)(z+x-y) &= cz, & (y+z-x)(z+x-y) &= cxy. \end{aligned}$$

Solution by T. WOODCOCK, B.A.; E. RUTTER; and others.

1. Substituting $p = y + z - x$, $q = \&c.$, $r = \&c.$, we get $2qr = a(q+r)$, $\&c. \&c.$, whence $\frac{1}{p} = \frac{1}{b} + \frac{1}{c} - \frac{1}{a}$; $\frac{1}{q} = \&c.$; $\frac{1}{r} = \&c.$ Therefore

$$z = \frac{q+r}{2} = \frac{1}{a} + \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c} \right) \left(\frac{1}{a} + \frac{1}{c} - \frac{1}{b} \right); \quad y = \&c., \quad z = \&c.$$

2. Substituting $lx = y+z-x$, $my = \&c.$, $nz = \&c.$, so that

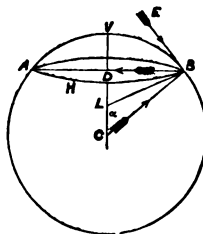
$$\begin{vmatrix} l+1, & -1, & -1 \\ -1, & m+1, & -1 \\ -1, & -1, & n+1 \end{vmatrix} = 0,$$

we get $mn = a$, $nl = b$, $lm = c$; whence $al = bm = cn = (abc)^{\frac{1}{2}}$, and, substituting in the determinant, we have $(abc)^{\frac{1}{2}} + a + b + c = 4$.

7007. (By Professor HUDSON, M.A.)—An endless elastic ring, natural length $2\pi l$, rests in a horizontal circle on a rough sphere, radius a , coefficient of friction $\tan \gamma$, and is just on the point of slipping when stretched to double its natural length: prove that, if the weight be neglected, $2l = a \cos \gamma$; and if the weight be w , $w \tan(a + \gamma) = 2\pi\lambda$, where a is the angular radius of the small circle formed by the string, and λ is the modulus of elasticity.

Solution by W. H. BLYTHE, M.A.; Dr. CURTIS; and others.

Let C be the centre of the sphere AVB, AHB the circle formed by the stretched ring, and r (=DB) the radius of this circle, the two tensions, T, at any point, B for example, inclined at the angle of contingence, compound at that point a resultant $Td\phi$, where ϕ is the angle of contingence, along BA, which is in equilibrium with the resistance of the surface along CB, and the friction along EB, the tangent is B:—(1) *alone* when the weight of the ring is neglected; (2) *compounded* with the weight of the element at B, and extended to the length of the element, when the weight of the ring is considered. In case (1) it is necessary that $\angle ABC = \gamma$ (the $\angle$ of friction), therefore $a \cos \gamma = r = 2l$; in case (2), if DL represent the weight of the element, while BD represents the force in the direction of r , extended to the element ds , BL must be inclined to BC at an angle $= \gamma$, therefore

$$\tan(a + \gamma) = \frac{BD}{DL} = \frac{Pds}{\frac{w}{2\pi r} ds}, \quad \therefore w \tan(a + \gamma) = 2\pi r P = 2\pi r \frac{T}{r} = 2\pi T,$$


but $T = \lambda \frac{\text{extension}}{\text{original length}} = \lambda$, therefore $w \tan(a + \gamma) = 2\pi\lambda$.

[More generally, if the length of the ring when stretched be to its length when unstretched as $K+1:1$, the conditions will be $a \cos \gamma = (K+1)l$, and $w \tan(a + \gamma) = 2\pi K\lambda$.]

$$\begin{aligned}\bar{x}' &= l\bar{x} + m\bar{y} + n\bar{z} = l\Sigma(a_1x_1) + m\Sigma(a_1y_1) + n\Sigma(a_1z_1) \\ &= \Sigma[a_1(lx_1 + my_1 + nz_1)] = \Sigma(a_1x_1'),\end{aligned}$$

which proves (1). Again, if $x'' = x' + h$, (2) follows thus:—

$$\bar{x}'' = \bar{x}' + h = \Sigma(a_1x_1') + h = \Sigma[a_1(x' + h)] \text{ if } \Sigma a_1 = 1 = \Sigma(a_1x_1'').$$

[This is the explanation of the form of the coordinates of the various centres in Mechanics.]

6843. (By W. J. C. SHARP, M.A.)—If a particle be free to move in a fixed circular tube, centre O, and if the particle be subject to the action of two forces, one repelling and the other attracting, each varying as (distance) $^{-n}$, and directed to two points F, F' which are reciprocal with respect to the circle; the particle will rest at any point of the tube if the forces at the unit of distance are as $(OF)^{\frac{1}{n-1}} : (OF')^{\frac{1}{n-1}}$. [Suggested by Question 4694.]

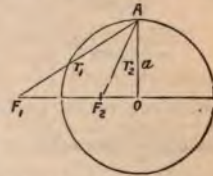
Solution by A. H. CURTIS, LL.D., D.Sc.; G. M. REEVES, M.A.; and others.

The conditions of the question give, as the surfaces of equilibrium

$$\frac{\mu_1 dr_1}{r_1^n} - \frac{\mu_2 dr_2}{r_2^n} = 0,$$

or
$$\frac{\mu_1}{r_1^{n-1}} - \frac{\mu_2}{r_2^{n-1}} = \text{const.} \dots\dots\dots (1).$$

Now, if the circular groove be represented in the figure, a being its radius and O its centre, and r_1, r_2 be the distances of any point A on it from the centres of force F_1 and F_2 , as



$$F_1O \cdot F_2O = a^2, \quad F_1O : OA = OA : F_2O,$$

therefore triangles OAF₁, OF₂A are similar, therefore $r_1 : OF_1 = r_2 : a$ or

$$OF_1 = \frac{ar_1}{r_2}; \text{ similarly } OF_2 = \frac{ar_2}{r_1}, \text{ therefore } \left(\frac{r_1}{r_2}\right)^2 = \frac{OF_1}{OF_2}, \text{ and}$$

$$\left(\frac{r_1}{r_2}\right)^{n-1} = \left(\frac{OF_1}{OF_2}\right)^{\frac{1}{2}(n-1)},$$

which can be identical with (1) by making the constant = zero,

provided $\frac{\mu_1}{\mu_2} = \left(\frac{OF_1}{OF_2}\right)^{\frac{1}{2}(n-1)}$; that condition be fulfilled, one

of the surfaces of equilibrium is obtained by rotating the circle whose centre is O, about O, and every section of this sphere is a curve of equilibrium.

6718.
right angle

triangle, O the
coincident with

the hypotenuse; find (α) the locus of their centroids, and show (β) that the normal to the locus at C meets the curve again in the centroid of the triangle.

Solution by R. KNOWLES, B.A., L.C.P.; E. RUTTER; and others.

Let PQNM be one of the rhombuses; then we have the following equations:— $x_1 + x_2 = 2x$, $2y = y_1 \dots (1, 2)$,

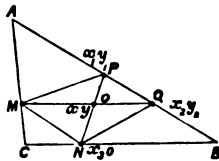
$$x_2 = 2x, \quad \frac{b}{a} = \frac{y_1}{a-x_1} = \frac{y_2}{a-x_2} \dots (3, 4),$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = y_2^2 + (x_2 - x_3)^2 \dots (5);$$

and, eliminating x_1, y_1, x_2, y_2, x_3 , we have (α) for the locus, the hyperbola about ABC,

$$abx^2 + 2c^2xy + aby^2 - a^2bx - ab^2y = 0 \dots (6).$$

The equation to the normal through C is $ay = bx \dots (7)$; substituting (7) in (6), we have $x = \frac{1}{2}a$, which proves (β).



7008. (By Professor MALET, M.A.)—If the solutions of the linear differential equations $\frac{d^2y}{dx^2} + Py = 0$, $\frac{d^2y}{dx^2} + Qy = 0$, be respectively $y = C_1y_1 + C_2y_2$, $y = C_3y_3 + C_4y_4$, where y_1, y_2, y_3, y_4 are functions of x , and C_1, C_2, C_3, C_4 arbitrary constants; prove that, Δ being a constant,

$$\Delta \equiv \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} & \frac{dy_4}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} & \frac{d^2y_4}{dx^2} \\ \frac{d^3y_1}{dx^3} & \frac{d^3y_2}{dx^3} & \frac{d^3y_3}{dx^3} & \frac{d^3y_4}{dx^3} \end{vmatrix} = \Delta (P - Q)^2.$$

Solution by Prof. WOLSTENHOLME, M.A.; W. H. BLYTHE, M.A.; and others.

$$\frac{d^2y_1}{dx^2} + Py_1 = 0, \quad \frac{d^2y_2}{dx^2} + Qy_2 = 0, \quad \frac{d^2y_3}{dx^2} + P \frac{dy_1}{dx} + y_1 \frac{dP}{dx} = 0, \text{ \&c.}$$

$$\therefore \Delta = - \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ Py_1 & Py_2 & Qy_3 & Qy_4 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} & \frac{dy_4}{dx} \\ P \frac{dy_1}{dx}, P \frac{dy_2}{dx}, Q \frac{dy_3}{dx}, Q \frac{dy_4}{dx} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ Py_1 & Py_2 & Qy_3 & Qy_4 \\ y_1 \frac{dP}{dx} & y_2 \frac{dP}{dx} & y_3 \frac{dQ}{dx} & y_4 \frac{dQ}{dx} \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} & \frac{dy_4}{dx} \end{vmatrix}$$

The second of these determinants is easily proved zero; and the first equals

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ 0, & 0, & (Q-P)y_3 & (Q-P)y_4 \\ \frac{dy_1}{dx}, & \frac{dy_2}{dx}, & \frac{dy_3}{dx} & \frac{dy_4}{dx} \\ 0, & 0, & (Q-P)\frac{dy_3}{dx}, & (Q-P)\frac{dy_4}{dx} \end{vmatrix} = -(Q-P)^2 \left(y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} \right) \times \left(y_3 \frac{dy_4}{dx} - y_4 \frac{dy_3}{dx} \right).$$

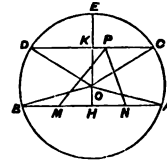
But $y_2 \frac{d^2 y_1}{dx^2} - y_1 \frac{d^2 y_2}{dx^2} = 0$, whence $y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} = C$,

and similarly $y_4 \frac{dy_3}{dx} - y_3 \frac{dy_4}{dx} = C'$, or $D = -CC'(Q-P)^2$.

6963. (By the Rev. H. G. DAY, M.A.)—If three points be taken at random within a circle of radius c , prove that the mean area of the triangle whose corners are the random points is $\frac{35c^2}{48\pi}$.

Solution by DAVID EDWARDS; Prof. SEITZ, M.A.; and others.

Let O be the centre of the circle, M, N, P the three random points, AB the chord through M, N , and CD the chord through P parallel to AB . Draw EKO perpendicular to AB . Let $OA = c$, $AM = x$, $MN = y$, $AB = x'$, $\angle AOE = \theta$, $\angle COE = \phi$, and ψ = the angle OE makes with a fixed radius.



Then $HK = c(\cos \phi - \cos \theta)$, $CD = 2c \sin \phi$, $x' = 2c \sin \theta$, and area $MNP = \frac{1}{2} MN \cdot HK = \frac{1}{2} cy(\cos \phi - \cos \theta)$. An element of the circle at M is $c \sin \theta d\theta dx$, at N it is $y dy d\psi$, and at P it is $2c^2 \sin^2 \phi d\phi$. The limits of θ are 0 and π ; of ϕ , 0 and θ , and doubled, since P may lie on either side of MN ; of x , 0 and x' ; of y , 0 and x , and doubled; and of ψ , 0 and π . Hence, since the whole number of ways the three points can be taken is $\pi^3 c^6$, we have for the average area of the triangle

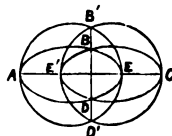
$$\begin{aligned} & \frac{4}{\pi^3 c^6} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^\pi \int_0^\pi \frac{1}{2} cy(\cos \phi - \cos \theta) c \sin \theta d\theta \cdot 2c^2 \sin^2 \phi d\phi dx y dy d\psi \\ &= \frac{4}{\pi^2 c^2} \int_0^\pi \int_0^\theta \int_0^{x'} \int_0^\pi (\cos \phi - \cos \theta) \sin \theta d\theta \sin^2 \phi d\phi dx y dy \\ &= \frac{16c^2}{3\pi^2} \int_0^\pi \int_0^\theta (\cos \phi - \cos \theta) \sin^5 \theta d\theta \sin^2 \phi d\phi \\ &= \frac{8c^2}{9\pi^2} \int_0^\pi (3 \sin \theta - \sin^3 \theta - 3\theta \cos \theta) \sin^5 \theta d\theta = \frac{35c^2}{48\pi}. \end{aligned}$$

[For other solutions see *Reprint*, Vol. 9., pp. 96, 97, and 34, pp. 42, 43.]

3246. (By Professor SIVERLY, M.A.)—Two equal ellipses, whose axes are in the ratio of m to n , and transverse axes in the same straight line, and the distance between their centres d , intersect one another; find their axes when the three spaces formed by the intersecting ellipses are equal.

Solution by A. MARTIN, M.A.; BELLE EASTON; and others.

Let ABCD be the two equal ellipses cutting one another in B and D. Around the centres of the ellipses, with radii equal to the semi-transverse axes, draw two equal circles, cutting one another in BD produced to B' and D'. Let the transverse axis of each ellipse be unity; then its area will be $= \frac{1}{2} \frac{n}{m} \pi$. It is clear, when the areas of the



three spaces are equal, the area of the segment BED or BE'D is $\frac{1}{2}$ the area of each ellipse $= \frac{1}{18} \frac{n}{m} \pi$.

But area of elliptic segment BED $= \frac{n}{m}$ (area circular segment B'ED').

Hence area circular segment B'ED $= \frac{1}{18} \pi = .1963495$.

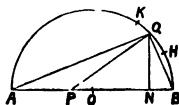
The versed-sine corresponding to this area is found by the table of circular segments to be $.298\frac{1}{8}$. And $(.5 - .298\frac{1}{8}) \times 2 = .403\frac{3}{4}$ distance between the centres of the supposed ellipses. And, since the sides of similar figures are proportional, we have $.403\frac{3}{4} : 1 = d : 2.47626d = \text{transverse axis required}$. The result shows that, if two equal ellipses be drawn as required with distance between their centres $= d$, and transverse axes $= 2.47626d$, and any conjugate axis whatever, the three spaces formed by the intersecting ellipses are equal.

6071. (By Professor MATZ, M.A.)—A point is taken at random in the diameter of a semicircle, a second point taken at random on the curve, and these points are joined with one end of the diameter; prove (1) that the mean area of the triangle thus formed is $\frac{r^2}{\pi}$; and (2) find the chance that one of these triangles is less than $\frac{1}{2} \pi r^2$.

Solution by G. HEFFEL, M.A.; KATE GALE; and others.

1. Let O be the centre; P the point in the diameter AB; Q the point in the semicircle; and QN a perpendicular on AB; then, wherever P is placed, the mean of the two triangles AQP, BQP is $\frac{1}{2} QN \cdot AB$ or $\frac{1}{2} r^2 \sin \theta$; therefore mean area of all the triangles

$$= \frac{1}{2} r^2 \int_0^{\frac{1}{2}\pi} \sin \theta \, d\theta + \frac{1}{2} \pi = \frac{r^2}{\pi}$$



Let $\angle H = \theta$ and $\angle K = \phi$ then the circular measures of BH and BK are $\frac{1}{2} \pi - \theta$ and $\frac{1}{2} \pi - \phi$ respectively. Let $\angle BOK = \alpha$, $\angle BOK = \beta$, $\angle BOK = \gamma$. Then, if $\theta < \alpha$, $\angle HOK$ must be $< \frac{1}{2} \pi$; if $\theta > \beta$, $\angle HOK$ must be $> \frac{1}{2} \pi$, supposing P to be somewhere in AO .

Let $HP = x$, and let $\theta = \frac{1}{2} \pi - \phi$, therefore $x = \frac{r \theta}{\frac{1}{2} \pi - \phi}$, and chance of θ being $< \frac{1}{2} \pi - \phi = \frac{2\theta}{\pi} = \frac{2}{\pi} \sin \theta$; therefore sum of probabilities between the limits x and 3

$$= \frac{2}{\pi} \int_0^{\frac{1}{2} \pi - \phi} \sin \theta d\theta = \frac{2}{\pi} [\log \tan \frac{1}{2} \theta]_0^{\frac{1}{2} \pi - \phi} = \beta + \alpha;$$

$$\text{Therefore total chance} = \frac{2}{\pi} [\log \tan \frac{1}{2} \theta - \log \tan \frac{1}{2} \phi] - \beta + 2\alpha + \frac{1}{2} \pi \\ = \frac{2}{\pi} [\log \tan \frac{1}{2} \theta - \log \tan \frac{1}{2} \phi] - \frac{2\beta - 4\alpha}{\pi}.$$

We find here $\beta = 1.77$, $\alpha = 2.37$; and the required probability = .476.

6781. (By R. TUCKER, M.A.)—B, C, D are three points on an ellipse, of centre O, the osculating circles at which intersect in a point A on the curve; tangents at B, C, D intersect in B', C', D'; prove that these points lie on a conic whose axes $2a$, $2b$, and are situated with regard to A' the point where AO meets the new ellipse) in like manner as A, B, C, D are.

Solution by D. EDWARDS: G. F. WALKER, M.A.; and others.

The points B, C, D form a maximum triangle inscribed in the ellipse, the tangent at B being parallel to CD, &c., and O being its centroid. Also the circle round BCD passes through A.

B, C, D are the middle points of the sides of B', C', D', and O is an inverse centre of similitude of the two figures, the ratio being 2 : 1; therefore, &c.

[The normals at B, C, D meet in a point, as likewise do those at B', C', D', and these two points lie on the normals drawn at the other end of the diameter through A (or A'); also, the area of the triangle BCD is $\frac{2}{3} \sqrt{3} \cdot ab$, and $\Sigma (OB^2) = \frac{2}{3} (a^2 + b^2)$.]

5099. (By R. F. SCOTT, M.A.)—If $2a$, $2b$, $2c$ are the axes of an ellipsoid, d the distance from the centre to any point on the surface, and p the perpendicular from the centre on the tangent at that point, prove that the length of the normal chord at the point is

$$2a^2b^2c^2 + p [b^2c^2 + c^2a^2 + a^2b^2 - p^2(a^2 + b^2 + c^2) + d^2p^2].$$

Solution by J. HAMMOND, M.A. ; Prof. MATZ, M.A. ; and others.

Let the coordinates of the point through which the chord is drawn be $(\lambda a, \mu b, \nu c)$, then we have

$$\lambda^2 + \mu^2 + \nu^2 = 1, \quad \lambda^2 a^2 + \mu^2 b^2 + \nu^2 c^2 = a^2, \quad \frac{\lambda^2}{a^2} + \frac{\mu^2}{b^2} + \frac{\nu^2}{c^2} = \frac{1}{p^2} \dots (1, 2, 3).$$

If, moreover, ρ denote the length of the normal chord, and (x, y, z) the coordinates of its other extremity, we have

$$\frac{\lambda a - x}{a} = \frac{\mu b - y}{b} = \frac{\nu c - z}{c} = p\rho, \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

giving
$$\lambda^2 \left(1 - \frac{p\rho}{a^2}\right)^2 + \mu^2 \left(1 - \frac{p\rho}{b^2}\right)^2 + \nu^2 \left(1 - \frac{p\rho}{c^2}\right)^2 = 1.$$

Reducing by means of (1) and (3),

$$\frac{\lambda^2}{a^4} + \frac{\mu^2}{b^4} + \frac{\nu^2}{c^4} = \frac{2}{p^2 \rho} \dots (4),$$

and eliminating λ^2, μ^2, ν^2 from (1), (2), (3), (4), we have the result in the form

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^2 & a^2 & b^2 & c^2 \\ \frac{1}{p^2} & \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \\ \frac{2}{p^2 \rho} & \frac{1}{a^4} & \frac{1}{b^4} & \frac{1}{c^4} \end{vmatrix} = 0.$$

To reduce this, multiply the last row but one by $\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$ and subtract it from the last, then the second row by $\frac{1}{a^2 b^2 c^2}$, also subtracted from the last, and the first row by $\left(\frac{1}{b^2 c^2} + \frac{1}{c^2 a^2} + \frac{1}{a^2 b^2}\right)$ added to the last; this will make all the elements of the last row vanish except the first, which becomes $\frac{2}{p^2 \rho} - \frac{1}{p^2} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \frac{a^2}{a^2 b^2 c^2} + \left(\frac{1}{b^2 c^2} + \frac{1}{c^2 a^2} + \frac{1}{a^2 b^2}\right)$, and the result given in the question follows directly.

6584. (By R. KNOWLES, B.A., L.C.P.)—Show that the locus of the centres of circles that pass through a given point, and touch a given straight line, is a parabola.

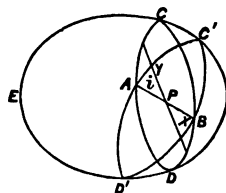
Solution by W. M. MEE, B.A. ; C. BICKERDIKE ; and others.

Let S be the fixed point, and XX' the given straight line. Then, if P be the centre of any of the circles, and PM be drawn perpendicular to XX', we have obviously SP = PM. Therefore the locus of P is a parabola, of which S is the focus and XX' the directrix.

6980. (By Professor MALLET, M.A.)—If through a point P, situated within the surface of a given ellipsoid, a plane be drawn so as to cut the volume of the ellipsoid most unequally, prove that the plane is conjugate to the line joining P to the centre of the ellipsoid.

Solution by Dr. CURTIS; J. HAMMOND, M.A.; and others.

Let ADBC be the section of the ellipsoid made by any plane through P cutting off the volume EADBC maximum (equal V suppose), and therefore the remainder of the ellipsoid minimum; let EAD'BC' be the section made by any other plane through P inclined to the former plane at the infinitesimal angle $\delta\theta$; the volume V being maximum, we must have $\delta V = 0$, and therefore the wedge ABCC' = the wedge ABDD'. Now, taking as axis of X AB, the intersection of the two cutting planes, and as axis of Y that diameter of the section ADBC which bisects the ordinate AB, and denoting by i the angle at which X and Y intersect, the expression for the wedge ABCC' will be $\delta\theta \iint y \sin^2 i \, dx \, dy$, or $\delta\theta \sin^2 i \iint y \, dx \, dy$, the integral being extended to the area ACB, while that for the wedge ABDD' will be the same extended to the area ADB. Denoting therefore these areas by a, a' , and equating the two integrals, we get $a\bar{y} = a'\bar{y}'$; consequently, the centre of gravity of the area $a + a'$, that is, of the ellipse ADBC, is on the line AB, and on every such line, it therefore coincides with the intersection of all such, viz., with the point P; this point is therefore the centre of the section, and therefore the line joining the centre of the ellipsoid to this point is the diameter conjugate to the section. As this section makes one segment of the ellipsoidal volume maximum, and the other minimum, the difference between the segments is maximum.



[See Prof. TOWNSEND's paper "On the Equilibrium of a Floating Body," in the *Cambridge and Dublin Mathematical Journal*, for May, 1849.]

6792. (By C. MORGAN, B.A.)—DAE is a triangle, having a right angle at A. From A lines AB, AC are drawn, making the angles DAB, DAC each equal to the angle E, and meeting DE in B and C respectively, show that $BA^2 : BE^2 = DC : EC$.

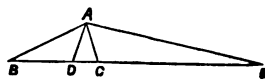
Solution by W. B. GROVE, B.A.; R. KNOWLES, B.A., L.C.P.; and others.

Since DA and EA bisect the angle BAC and its supplement, therefore

$$BE : EC = BD : DC.$$

But $BE : BA = BA : BD$,

therefore $BE^2 : BA^2 = BE : BD = EC : DC$.



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VOL. XXXVIII.

LONDON:

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CONTENTS.

Solved Questions.

4622. (S. Tebay, B.A.)—A fixed pulley rests on a smooth horizontal plane, and a string is fastened to its rim, and a weight (P) to the other end of the string, which lies on the plane, the string being tangential to the pulley. If the weight receives a given impulse perpendicular to the string, determine the motion, and show that there is an epoch at which the string ceases to coil round the pulley. Assign the subsequent motion of P.....Page 113
4784. (H. Hart.)—If ABC be a spherical triangle, and AT the tangent arc to the circumscribing small circle, prove that the angle BAT is equal to the angle C in the alternate segment minus half the spherical excess of the triangle. 45
4981. (S. Tebay, B.A.)—A circular disc can revolve freely in a vertical plane about its axis, while a heavy particle descends in a curvilinear groove between two given points in its plane; find the brachystochrone, and assign the limits of possibility..... 106
5149. (S. Tebay, B.A.)—A solid, having a plane base, rests on a smooth horizontal plane, and a heavy particle descends in a vertical plane passing through the centre of gravity in a curvilinear groove between two given points in the solid; find the brachystochrone..... 43
5490. (S. Tebay, B.A.)—AB, AC are two straight rods in a vertical plane, on which are two rings, B, C held in equilibrium by a flexible string passing through them, and having its ends fixed at A. If the equilibrium be slightly disturbed, find the time of a small oscillation. 105
5615. (S. Tebay, B.A.)—In the arbelon of the ancient geometers, three circular segments are drawn on the same straight line (length 6), and on given parts of it (4 and 2). Given the sum of the three arcs, find the radii when the sum of the areas of the segments is a maximum. 66
5622. (J. L. McKenzie, B.A.)—If s_r denote the sum of the r^{th} powers of the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} \dots + p_n = 0$; find the value of the determinant

s_0	s_1		s_{i-1}	s_{i+1}		s_n	
s_1	s_2		s_i	s_{i+2}		s_{n+1}	
...	...	...	...	...	...	...	
s_{k-1}	s_k		s_{k+i-2}	s_{k+i}		s_{n+k}	
s_{k+1}	s_{k+2}		s_{k+i}	s_{k+i-2}		s_{n+k+1}	
...	...	...	...	...	...	...	
s_n	s_{n+1}		s_{n+i-1}	s_{n+i+1}		s_{2n}	

..... 86

5710. (E. W. Symons, M.A.)—If l_1, l_2, l_3, l_4 (of which l_4 is the greatest) be the semi-parameters of the four conics circumscribing a triangle, each having one focus at the orthocentre, prove that

$$\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} > \frac{p_1 + p_2 + p_3}{\rho_2} > \frac{9}{l_4},$$

p_1, p_2, p_3 being the perpendiculars of the triangle, and ρ the radius of its self-conjugate circle. 107

5740. (Professor Wolstenholme, M.A.)—If $a > b > c > d$, prove that

$$\int_b^a \left[\frac{dx}{(a-x)(x-b)(x-c)(x-d)} \left\{ \frac{(a-x)(x-d)}{a-d} + \frac{(x-b)(x-c)}{b-c} \right\} \right] \\ - \int_d^c \left[\frac{dx}{(a-x)(b-x)(c-x)(x-d)} \left\{ \frac{(a-x)(x-d)}{a-d} + \frac{(b-x)(c-x)}{b-c} \right\} \right]^{\frac{1}{2}}. \quad 93$$

5843. (Rev. T. R. Terry, M.A.)—If the coordinates of the points A, B, C, D, be represented by (a_1, a_2, a_3) , and so on, prove that the equation to the tangent plane at A to the sphere which passes through A, B, C, D, is

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ x-a_1 & y-a_2 & z-a_3 \end{vmatrix} = P [\mu_1 (x-a_1) + \mu_2 (y-a_2) + \mu_3 (z-a_3)] \\ + Q [\lambda_1 (x-a_1) + \lambda_2 (y-a_2) + \lambda_3 (z-a_3)],$$

where $P = (b_1 - a_1)(c_1 - b_1) + (b_2 - a_2)(c_2 - b_2) + (b_3 - a_3)(c_3 - b_3)$,

$$Q = (d_1 - c_1)(a_1 - d_1) + \&c.$$

$$\lambda_1 = (b_2 - a_2)(c_3 - b_3) - (b_3 - a_3)(c_2 - b_2), \lambda_2 = \&c., \lambda_3 = \&c.,$$

$$\mu_1 = (d_2 - c_2)(a_3 - d_3) - (d_3 - c_3)(a_2 - d_2), \mu_2 = \&c., \mu_3 = \&c. \dots 99$$

5894. (Professor Seitz, M.A.)—A circle is divided into two semi-circles, one of which is divided into two quadrants. If a point is taken at random in the surface of each quadrant, find the chance that the chord drawn through the points will be less than a line of given length. 110

5905. (Professor Cochez.)—Quatre circonférences étant données, trouver sur l'une d'elles un point tel que les polaires de ce point par rapport aux trois autres circonférences le coupent au même point. 22

6079. (S. Tebay, B.A.)—The late Samuel Bills has stated (*Reprint*, Vol. XXII., p. 71) that "any number can be resolved into three rational cubes." Is this an established fact? Show that the equation $x^3 + y^3 + z^3 = 2\mu^3$ admits of a general solution; and thence find three numbers whose sum is unity, and such that, if each be taken from unity, the remainders shall be cubes. 101

6129. (Professor Matz, M.A.)—Two points are taken at random in the arc of a given circular quadrant of radius 1; and a third point is taken at random, (1) in one of the rectilinear sides of the quadrant, and (2) in the surface of the adjacent quadrant; and these are joined, in their respective order, by straight lines; prove that the mean areas of the triangles thus formed are $\Delta_1 = \frac{r^2}{\pi^2} (3\pi - 8)$, $\Delta_2 = \frac{r^2}{\pi^2} (2\pi - 4)$ 24

6135. (W. S. B. Woolhouse, F.R.A.S.)—Find (1) the average area of triangles drawn on the surface of a given rectangle, and having one of the three sides parallel to a given line; and (2) the directions of the given line when the average area is a maximum and a minimum. 29

6149. (S. Tebay, B.A.) — Required a direct general solution of the equation $N^2 = H + 3et(2m-s-t)$, without assuming particular values of s, t to make the absolute term a square. [This problem solves the celebrated diophantine problem,—“To find n numbers such that, if each be taken from the cube of their sum, the remainders shall be cubes.” The results are $24r^2(3mr^2+1)s = 9Hr^6-4, (3r^2s+1)t = 2m-s.$] ... 81
6159. (Professor Cochez.)—Un paraboloïde elliptique est coupé par un plan perpendiculaire à l'axe. Trouver le plus grand parallépipède rectangle que l'on puisse inscrire dans la portion de paraboloïde déterminée par le plan sécant. 60
6164. (Professor Seitz, M.A.)—If A, B, C, D, E are five random points within a sphere, prove that the chance that any one of the points is within the tetrahedron having the other four points for its vertices is $\frac{2}{15\pi}$ 62
6169. (Elizabeth Blackwood.)—If A, B, C are random points within a sphere of which O is the centre and r the radius, prove that the average volume of the tetrahedron ABCO is $\frac{1}{105\pi} \pi r^3$ 62
6182. (D. Edwardes.) — The ends of a uniform rod of length $2a$ slide upon a smooth vertical circle (without inertia) of radius $\frac{2}{3}a\sqrt{3}$. If the system be set rotating about the fixed vertical diameter, prove that the inclination of the rod to the vertical at any time is given by the equation $\left(\frac{d\theta}{dt}\right)^2 = \frac{g\sqrt{3}}{a}(\sin\theta - \sin\alpha)$, α being the initial value of θ ... 108
6194. (Professor Matz, M.A.)—Three unequal homogeneous spheres being thrown into a given hemispherical bowl; determine their position when in equilibrium. 94
6195. (Professor Cochez.) — Démontrer que le lieu des foyers des hyperboles ayant un sommet donnée et une asymptote également donnée est une Strophoïde droite. 49
6250. (Professor Seitz, M.A.)—A, B, C are random points within a hemisphere; find the chance that the plane through A, B, C does not intersect the base of the hemisphere. 118
6412. (Rev. T. P. Kirkman, M.A., F.R.S.) — The substitution 56781234 has twelve square roots, each having two circular factors of four elements. The powers and products of these twelve θ, ϕ , &c., viz., $\theta^2, \theta\phi, \phi\theta$, &c., are a transitive group G of 32 substitutions, of which the title is $8.4 = 1 + 12_{44} + 13_{2222} + 6_{221111}$; $Q = 105$: i.e., the group contains besides unity 12345678 and the 12, θ, ϕ , &c., 13 square roots of unity having no element undisturbed, and six square roots of unity having four undisturbed elements. The group of 32 has, including G, 105 equivalents, $G, \alpha G\alpha^{-1}, \beta G\beta^{-1}$, &c. Construct a transitive group H of the same order 32, having the same title, except that the number of its equivalents is $Q = 630$, in which the substitutions 12_{44} are not all square roots of the same substitution of the second order. [The interest of the question lies in the fact that this is the simplest of all cases of two transitive groups whose titles differ in nothing except the number Q.] 80
6425. (W. J. C. Sharp, M.A.)—Prove that one and only one quadric surface can be drawn through a given twisted quintic curve. 36
6568. (Professor Seitz, M.A.)—A quadrilateral is formed by joining

the ends of two chords, each of which is drawn at random through a random point within a circle; show that the average area of the quadrilateral is $\frac{17r^2}{4\pi}$ 77

6604. (Professor Matz, M.A.)—Prove that the mean area of the maximum elliptic disc placed upon the surface common to two equal circles of radius unity, that are thrown upon each other at random, is $\frac{1}{2}\pi(3\pi-8)$ 68

6619. (W. J. C. Sharp, M.A.)—With the notation adopted in Question 6493, prove that

$$\begin{aligned} \frac{(p^{i+1}-1)(p^{i+2}-1)\dots(p^{i+j}-1)}{(p-1)(p^2-1)\dots(p^j-1)} &= \frac{p^{ij+1}-1}{p^j-1} x_{ij} + \frac{p^{ij}-1}{p-1} x_{ij-1} \\ &+ p \frac{p^{ij-1}-1}{p-1} x_{ij-2} + p^2 \frac{p^{ij-2}-1}{p-1} x_{ij-3} + p^3 \frac{p^{ij-3}-1}{p-1} x_{ij-4} + \&c. \end{aligned}$$

..... 116

6623. (Professor Hudson, M.A.)—If the angular radius of the small circle drawn round a triangle be $\tan^{-1} 2$, the area of the triangle be one-sixth of the sphere, and one side be a quadrant, find all the parts of the triangle..... 60

6625. (C. Leudesdorf, M.A.)—Equilateral triangles BDC, CEA, AFB are drawn externally on the sides of a plane triangle ABC; prove that, if AD, BE be joined, they will intersect at a point whose distance from AB is $\frac{4}{AB} \cdot \frac{(AFC)(BFC)}{3(ABC) + (BCD) + (CAE) + (ABF)}$, where (AFC) denotes the area of the triangle AFC, &c. 57

6644. (W. J. C. Sharp, M.A.)—When the catalecticant of a binary $2n$ -ic $(a, b, c \dots k, l)(x, y)^{2n}$ vanishes, prove that the n quantities to the sum of whose $2n^{\text{th}}$ powers the quantic is reducible are the factors of the canonizant of $(a, b, c \dots k)(x, y)^{2n-1}$ 57

6680. (Professor Mukhopādhyāy, M.A.)—Trisect a given triangle by straight lines at right angles to the base. 113

6683. (G. F. Walker, M.A.)—If n particles of masses $m_1, m_2 \dots m_n$ respectively, are rigidly connected together, and are capable of motion one on each of a series of smooth concentric circles in a vertical plane of radii $a_1, a_2 \dots a_n$ respectively; show that the time of a small oscillation about the position of stable equilibrium under the action of gravity is

$$2\pi \left(\frac{\sum m_r a_r^2}{g (\sum m_r^2 a_r^2 + 2S m_r m_s a_r a_s \cos \theta_{rs})} \right)^{\frac{1}{2}},$$

where Σ denotes summation from $r = 1$ to $r = n$, and S denotes summation for all different pairs of values of r and s , and θ_{rs} is the angle subtended at the common centre by the line joining m_r and m_s 46

6690. (J. Hammond, M.A.)—Prove that the limiting value, when $m = 0$, of $\frac{1}{\Gamma(m)} \int_0^1 e^{xy} \frac{dy}{(1-y)^{1-m}} = e^x$ 55

6709. (Dr. Macfarlane, D.Sc., F.R.S.E.)—A is a great-grandchild of B, B is a grandparent of C, and C is a child of D. What, expressed independently of B and C, is the relationship of A to D? 39

6733. (G. F. Walker, M.A.)—Prove that the result of eliminating the arbitrary function from $F(X_1, X_2, \dots, X_r, \dots, X_n) = 0$, where

$X_r = S^\lambda (x_r - u)$, $S = x_1 + \dots + x_n + u$, and $x_1 \dots x_n$, are independent variables, is

$$\sum_{r=1}^{r=n} [-\lambda(n+1)x_r + (1+\lambda)S] \frac{du}{dx_r} = -\lambda(n+1)u + (1+\lambda)S. \dots 31$$

6734. (H. G. Dawson.)—If $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ have a double root α , prove that $H = b^2 - ac$, $I = ae - 4bd + 3c^2$,

$$J = ace + 2bcd - ad^2 - eb^2 - c^3, \quad \alpha = -\frac{b}{a} + \left(\frac{H}{a^2} \pm \frac{3J}{aI}\right)^{\frac{1}{2}}. \dots 62$$

6741. (Professor Hoover.)—Show that the average area of circles whose diameters are the focal chords of an ellipse of semi-axes a, b , is

$$\frac{\pi b}{2a} (a^2 + b^2). \dots 47$$

6744. (Dr. Macfarlane, F.R.S.E.)—Deduce the necessary mathematical consequences of the law that a man cannot be the husband of his wife's sister. 27

6761. (G. F. Walker, M.A.)—A railway engine is drawing a train of equal carriages connected by spring couplings of strength μ , and the driving power is so adjusted that the velocity is $a + b \sin nt$. Show that, if n be nearly equal to $b \left\{ \frac{2\mu}{(M+4m)b^2 + 4m\kappa^2} \right\}^{\frac{1}{2}}$, the couplings will probably break, M being the mass of a carriage, which is supported on four equal wheels of mass m , radius b , and radius of gyration κ 67

6769. (Professor Townsend, F.R.S.)—Three forces in a common space being supposed transferred to the centre of the quadric determined by their three lines of direction in the space; show, on elementary principles, that

(a) The plane of the resultant moment is conjugate to the direction of the resultant force with respect to the surface;

(b) The pitch of the equivalent wrench $= abc \div r^2$; where a, b, c are the semi-axes of the surface, and r its radius in the direction of the force. 58

6777. (Dr. Macfarlane, F.R.S.E.)—Find all the meanings which the term *first cousin* may have in accordance with the English laws of marriage. 51

6782. (R. A. Roberts, M.A.)—In the motion of a rigid body about a fixed point under the action of no forces, show that the plane, containing two positions of the invariable line in the body which are separated by a constant interval of time, touches a cone concyclic with the invariable cone. 21

6784. (A. McIntosh, B.A.)—Prove that (1) from any point of a Folium of Descartes which is not situated on the loop, two real tangents can be drawn to the curve; (2) if P be the point, PT, PT' the tangents, and N the node, NT and NT' make equal angles with the nodal tangents; (3) if PN be produced on to meet the chord TT' , it is bisected in N , and it also bisects TT' ; (4) if the tangent (PQ) to the curve at the point P be produced on to meet the chord TT' in Q , PQ is bisected by the curve; (5) the chord TT' makes with the nodal tangents a triangle of constant area, and therefore touches a rectangular hyperbola having those tangents for asymptotes; (6) the rectangular hyperbola in (5) touches the curve at the vertex of the loop; (7) if the chord TT' meet the loop in a second

point R, PR being the line joining the primary point to this, PR also touches the rectangular hyperbola in (5); (8) the half of the chord TT' remote from the loop subtends the same angle as the segment RT' within the loop at the node; (9) the middle point of TT' lies on an identical folium having its loop in the opposite quadrant; (10) if from the middle point of TT' two tangents be drawn to the Folium in (9), the line joining the points of contact will pass through P; (11) the point of trisection of any tangent chord such as PT, most remote from the point of contact T, lies on a Folium of Descartes; (12) the centroid of the triangle PTT' is such a point; (13) in nodal cubics in general, NT and NT' form a harmonic pencil with the nodal tangents; (14) also TT' and PR touch a conic section which touches the nodal tangents. 82

6786. (Professor Mukhopādhyāy, M.A.)—Prove (1) geometrically that the rectangle contained by the perimeters of an acute-angled triangle and its pedal triangle, is equal to twice the sum of the rectangles contained by the perpendiculars, two and two; and (2) investigate whether the proposition holds for obtuse-angled triangles..... 47

6791. (E. W. Symons, M.A.)—A chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{(a^2 - b^2)^2}$; prove that the normals at its extremities meet on the ellipse. 57

6797. (Professor Townsend, F.R.S.)—In the irrotational strain of an incompressible substance in a tridimensional space, if the equipotential surfaces of the strain be the same as of the attraction for the ordinary law of the inverse square of the distance of a corresponding distribution of matter in the space, show that the potential of the strain will be in general of the same form as that of the attraction throughout the entire extent of the substance..... 70

6798. (Professor Wolstenholme, M.A.)—If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on an ellipse ($b^2x^2 + a^2y^2 = a^2b^2$), prove that (1) the area of the triangle whose corners are the vertices of the quadrangle is

$$\pm 4ab \frac{\sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\alpha - \delta) \sin \frac{1}{2}(\gamma - \alpha) \sin \frac{1}{2}(\beta - \delta) \sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta)}{\sin \frac{1}{2}(\beta + \gamma - \alpha - \delta) \sin \frac{1}{2}(\gamma + \alpha - \beta - \delta) \sin \frac{1}{2}(\alpha + \beta - \gamma - \delta)};$$

(2) this is also the area of the triangle formed by the diagonals of the quadrilateral formed by the tangents at the four points, the two triangles being identical; and (3) if the normals at the four points meet in one point (X, Y), the area of the triangle is

$$ab \frac{\{[a^2X^2 + b^2Y^2 - (a^2 - b^2)^2]^3 + 27a^2b^2(a^2 - b^2)^2 X^2Y^2\}^{\frac{1}{2}}}{(a^2 - b^2)(a^2X^2 - b^2Y^2)} \dots\dots\dots 69$$

6805. (A. McIntosh, M.A.)—In Question 6784 prove that (1) the line joining the points of contact of the conic in (14) with the nodal tangents, passes through the points of inflexion of the cubic; (2) if two imaginary tangents be drawn from the point R (i.e., the third point in which TT' meets the curve), prove that the chord joining the imaginary points of contact will pass through the primary point P; (3) the point in which PN meets TT' lies on another nodal cubic which has the same nodal tangents and the same conic as in (14, 6784); (4) if two real tangents be drawn from the point of intersection of PN and TT' to this latter cubic, the line joining the points of contact is the same as the line mentioned in (2), and therefore passes through P; (5) a line drawn through a fixed point on a nodal cubic intersects the curve again in two points; if two

pairs of tangents be drawn from these points to the curve, the chords joining their points of contact intersect on a fixed line passing through the node; (6) the point in which this fixed line meets the curve, and the line joining the first fixed points to the node, bear the same relation to one another; (7) in a cuspidal cubic, if the line joining any two points on the curve subtends a right angle at the cusp, it touches a conic which passes through the cusp, and cuts the cubic there at right angles. 24

6812. (Professor Mukhopādhyāy, M.A.)—Find the moment of inertia of (1) a circle, (2) a regular hexagon, in each case about an axis making an angle α with the plane of the figure. 100

6814. (Professor Hudson, M.A.)—Prove (1) the following construction for the centre of gravity of a right circular cone:—let AB be a diameter of the base, V the vertex, divide VA, VB in P, Q, in the ratio 3:2, where BP, AQ intersect in the centre of gravity; and show (2) that the same construction holds for a cone on an elliptic base. 36

6824. (Byomakesa Chakravarti, M.A.)—An infinite number of radii vectors are drawn from the centre of an ellipse of axes $2a, 2b$; if m be the mean value of the squares of these radii, when the differences of their successive excentric angles are equal, and m' the mean value of the squares when the successive angles between the radii themselves are equal, prove that $m + m' : m - m' = (a + b)^2 : (a - b)^2$ 64

6840. (W. M. Mee, M.A.)—If from any point on a fixed ordinate of a parabola, three normals be drawn to the curve, prove that the centroid of the triangle whose corners are the three points where the normals cut the curve, is a fixed point on the axis. 98

6841. (Rev. A. J. C. Allen, M.A.)—From a point O a straight line OA is drawn, making an angle $\alpha \left(< \frac{\pi}{n+1} \right)$ with a fixed straight line AB, and n other straight lines $OA_1, OA_2 \dots OA_n$ are drawn to it, making the angles $\angle OAA_1, \angle A_1OA_2 \dots$ all equal and each = α ; if $R_1, R_2 \dots R_n$ be the radii of the circles circumscribing the triangle OAA_1, OA_1A_2 ; prove that, if a be the perpendicular from O on AA_n , then

$$R_1 + R_2 + \dots + R_n = - \frac{a \sin na}{\sin 2\alpha \cos(n+1)\alpha}. \dots\dots\dots 45$$

6842. (G. F. Walker, M.A.)—A system of $2n$ convex lenses, of equal numerical focal length f are placed with their axes in the same straight line and their centres at a distance $4f$ apart, except the two middle ones, which are at a distance $8f$ apart; show that the focal length of a lens which must be placed midway between the two middle ones in order that the image of a bright point at a distance $4f$ in front of the first lens may be formed at an equal distance behind the last lens, is $\frac{2(n+1)}{2n+1} f$. 61

6852. (Professor Cavallin, M.A.)—Within a sphere a point P is taken at random, and two other points Q, R are taken at random, Q in the sphere concentric with the given sphere and passing through P, and R in the sphere concentric with the given sphere and touching PQ; prove that, if a denotes the radius of the given sphere, p the perpendicular from its centre on the plane determined by the points P, Q, R, and n any positive number, the average value of p^n is

$$\frac{3 \cdot 5 \cdot 7 \cdot 9}{(n+1)(n+3)(n+5)(n+7)(n+9)} a^n. \dots\dots\dots 35$$

6858. (Professor Wolstenholme, M.A.)—The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is cut at right angles by a concentric (not confocal) conic; prove that (1) the common chords which are not diameters touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2 - b^2};$$

(2) the locus of the intersection of normals to the hyperbola at the ends of such a common chord is the curve

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \frac{x^2 + y^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 - b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2,$$

whose asymptotes are $\frac{x}{a} \pm \frac{y}{b} \pm \left(\frac{a^2 + b^2}{a^2 - b^2}\right)^{\frac{1}{2}} = 0$ 84

6860. (Professor Matz, M.A.)—If the curve $\rho = m \sin 2\theta$ be inscribed in the ellipse $\rho^2 = b^2 + (1 - e^2 \cos^2 \theta)$; prove that

$$3e^2 \sin^2 \theta = (1 - e^2 + e^4)^{\frac{1}{2}} + 2e^2 - 1. \dots\dots\dots 52$$

6869. (A. McIntosh, M.A.)—Prove that (1) only two real tangents can be drawn from any point on the curve itself to a lemniscate; (2) the same line passes through the two real and the two imaginary points of contact of the four tangents that are analytically possible; (3) the following construction gives this line,—draw a radius-vector from the real node to the point from which the tangents are required to be drawn, produce it backwards one-half of its length, and through this extremity draw a line perpendicular to it, and this last will be the line sought; (4) hence a chord joining the points of contact of tangents drawn from any point on the curve touches an equilateral hyperbola, which has the same axis and the same foci as the lemniscate, which passes through the points of contact of the double tangents (which, of course, touches all the tangents that can be drawn from the three nodes to the curve); (5) to a lemniscate is drawn a tangent, which meets the curve again in two points, if a circle be drawn through the node and these two points, its centre lies on the lemniscate and its radius is equal to the radius vector from the node to the point of contact of the tangent; (6) if four tangents be drawn from any point (x', y', z') on the curve $ax^{-2} + by^{-2} + cz^{-2} = 0$ to touch the curve elsewhere, the four points of contact lie on the line $xx'^{-1} + yy'^{-1} + zz'^{-1} = 0$, which touches the conic $a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 = 0$, which passes through the eight points of contact of bitangents to the curve; (7) the above trinodal quartic includes the case of the lemniscate. 40

6886. (S. Constable, M.A.)—Prove that the distance between the centre of the circumscribing circle of a triangle and the orthocentre is

$$[9R^2 - (a^2 + b^2 + c^2)]^{\frac{1}{2}}. \dots\dots\dots 53$$

6902. (G. F. Walker, M.A.)—A series of quadrics $\frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} = 1$ is drawn through a fixed point (α, β, γ) : show that the locus of the centres of principal curvature at the fixed point is the surface

$$ayz(x - \alpha)^2 + \beta xz(y - \beta)^2 + \gamma xy(z - \gamma)^2 = 0. \dots\dots\dots 96$$

6905. (Professor Hudson, M.A.)—A heavy pulley, of weight R, hangs by a string over a fixed pulley, the other end of the string being attached to a weight W; over the pulley R hangs a string with unequal weights P, Q at its ends; find the accelerations of P, Q, W; and, as an example, let P = 5 lbs., Q = 4 lbs., R = 2 lbs., W = 12 lbs. 22

- 6913, 6940. (Rev. A. J. C. Allen, M.A., and J. Young, M.A.)—If K be the orthocentre of a triangle, P any point on the circumscribing circle, and if PK cut the pedal line of P in Q; prove that (1) PK is bisected in Q, (2) the locus of Q is the nine-point circle. 28
6916. (R. Tucker, M.A.)—ABC is a triangle, and PQR is an equilateral triangle in the circle ABC; prove that (1) the pedal lines, corresponding to the points P, Q, R, form another equilateral triangle; (2) the pedal lines, corresponding to points at equal arcual distances from any angle (A), intersect on a straight line. 29
6952. (Professor Townsend, F.R.S.)—Two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ being supposed to act in a common space; show that the complete locus of the entire system of points in the space for which their principal moments have similar or opposite directions, consists of three right lines, one situated at infinity in the space, and all three lying in planes parallel to the two central axes of the systems. 90
6957. (Professor Wolstenholme, M.A.)—Prove that
- $$\int_0^{\pi} \frac{\sin^2 nx}{\sin^2(1-n)x} dx = n \int_0^{\pi} \frac{\sin^2 x}{\sin^2(1-n)x} dx = \frac{n}{1-n} \int_0^{\pi} \frac{\sin(1+n)x}{\sin(1-n)x} dx \dots (1),$$
- $$\int_0^{\frac{\pi}{n}} \frac{\sin^2 nx}{\sin^2(1-n)x} dx = n \int_0^{\frac{\pi}{n}} \frac{\sin^2 x}{\sin^2(1-n)x} dx = \frac{n}{1-n} \int_0^{\frac{\pi}{n}} \frac{\sin(1+n)x}{\sin(1-n)x} dx \dots (2),$$
- if, in (1), $n > 0 < 2$, and in (2) $n > \frac{1}{2} < \infty$ 33
6962. (J. J. Walker, M.A.)—Show that the cosine of half the angle between tangents drawn from a point on an ellipse to an inner confocal ellipse, varies inversely as the diameter of the former parallel to its tangent at that point. 59
6968. (D. Edwardes.)—Prove that
- $$(y+z-2x')(z+x-2y')(x+y-2z') - (y+z-2x')(x+x'-y'-z')^2 - \dots \dots \dots$$
- $$\equiv 2(x+x'-y'-z')(y+y'-z'-x')(z+z'-x'-y'). \dots \dots \dots 65$$
6971. (G. S. Carr, B.A.)—If a number of voltaic cells differing both in electro-motive force and conductivity be joined "in multiple arc," show that the difference of potentials of the electrodes of the battery, when connected by a wire, is equal to the sum of the current strengths of the separate cells divided by the sum of the conductivities of all parts of the circuit. 104
6973. (Rev. T. W. Openshaw, M.A.)—On a focal chord of a parabola as diameter is described a circle cutting the parabola again in P, Q; prove that the circle PSQ will touch the parabola. 55
6974. (A. McMurchy, B.A.)—If $x+y+z=0$ and $a+b+c=0$, prove that $4(ax+by+cz)^3 - 3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) - 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x) = 54abcxyz \dots \dots \dots 96$
6975. (E. Rutter.)—If s_1 = sum of the odd terms, and s_2 = sum of the even terms, of $(a+b)^n$, show that $s_1^2 - s_2^2 = (a^2 - b^2)^n \dots \dots \dots 65$
7006. (Professor Nash, M.A.)—The reciprocal polar of a given central conic with respect to a circle is similar to the given conic; prove that (1) the locus of the centre of the circle is a bicircular quartic having a node at the centre, and foci coinciding with the foci of the conic; (2) if the given conic is an ellipse, the nodal tangents are the equi-conjugate dia-

meters; (3) if the conic is an hyperbola, the node is a conjugate point; (4) in the latter case the eccentricity of the hyperbola must be $< 2\frac{1}{2}$ for the locus to be a real curve. 48

7012. (J. J. Walker, M.A.)—Show that (1) the squares of tangents drawn from an external point (x', y') to an ellipse are, if $s' = b^2x'^2 + a^2y'^2 - a^2b^2$, given by the equation

$$(b^2x'^2 + a^2y'^2)^2 t^4 - 2[s'(x'^2 + y'^2) + b^4x'^2 + a^4y'^2]s't^2 + [c^4 - 2c^2(x'^2 - y'^2) + (x'^2 + y'^2)^2]s'^2 = 0;$$

(2) the lengths t, t' of the tangents are given linearly by

$$(b^2x'^2 + a^2y'^2)t^2 - \frac{2a'b'}{k}s't + (a^2 + b'^2 - x'^2 - y'^2)s' = 0;$$

$$\text{viz., } t + t' = \frac{2a'b'}{k}\sin^2 \frac{1}{2}(\theta - \theta'), \quad t - t' = \frac{c^2k}{a'b'}\sin(\theta + \theta')\tan \frac{1}{2}(\theta - \theta'),$$

where a', b' are the semi-axes of the confocal ellipse through (x', y') ; $k^2 = a'^2 - a^2 = b'^2 - b^2$; and θ, θ' are the excentric angles of the points of contact of the tangents t, t' respectively..... 44

7015. (W. J. C. Sharp, M.A.)—Show that (1) if the two conics $S = 0$ and $S' = 0$ have double contact, $F \equiv \lambda S + \mu S', \Phi \equiv \Lambda \Sigma + M \Sigma'$; if λ, μ have the same values as in (1), (2) the equations to the common tangents and common chord are

$$\lambda S - \mu S' = 0 \text{ and } \{(\mu \Delta)^{\frac{1}{2}} S - (\lambda \Delta)^{\frac{1}{2}} S'\}^{\frac{1}{2}} = 0;$$

(3) $\Lambda \Sigma - M \Sigma' = 0$ represents the points of contact; (4) $(\Lambda)^{\frac{1}{2}} \Delta' \Sigma - (M)^{\frac{1}{2}} \Delta \Sigma' = 0$ represents the intersection of the common tangents; also (5) furnish an answer to the query, in regard to Question 6204, on p. 75 of Vol. xxxiv. of the *Reprints*..... 32

7018. (W. R. Westropp Roberts, M.A.)—Determine the locus of the point of contact of a geodesic tangent on an ellipsoid, touching a fixed line of curvature, with a series of sphero-conics on the same surface.... 49

7021. (W. Nicholls, B.A.)—Let ABCD be any quadrilateral circumscribable by a circle, P the intersection of the sides AD, BC; L the middle point of CD; and M the point where PL produced meets AB: prove that $MA : MB = AP^2 : PB^2$ 66

7022. (Rev. H. G. Day, M.A.)—In a closed convex curve, if m be the average area of a triangle one of whose angular points is the centre G, the two others being taken at random in the area; m_1 the average area when all three are taken at random; k_1, k_2 the principal semi-axes of gyration; and S the area; prove that $m_1 = \frac{2}{3}m + \frac{k_1^2 k_2^2}{2S}$ 54

7023. (Rev. W. A. Whitworth, M.A.)—Four different rectangular parallelepipeds on square bases have their diagonals equal, and all their edges commensurable with the diagonal; find the solution in lowest integers. 56

7025. (W. A. Pick.)—Prove that, if a number of 3 or 4 figures is divisible by 7, half of the tens and units added to the other figure or figures will be divisible by 7..... 64

7032. (Professor Johnson, M.A.)—If three triangles have, when taken in pairs, a common axis of homology, prove that (1) the three centres of homology are in a straight line; (2) reciprocally, if three triangles have, when taken in pairs, a common centre of homology, the three axes of homology pass through a common point. 40

7039. (T. C. Simmons, M.A.)—If two asymptotes of a cubic meet on the curve at a point of inflexion, prove that the third asymptote will pass through the same point..... 95

7041. (W. J. C. Sharp, M.A.)—Find the form of the catenary in which an extensible string, uniform when unstretched, will hang on the centre of gravity. 119

7043. (J. Hammond, M.A.) — A, B, C, D are four points on a cubic curve, BC meets the curve again in A', CA in B', AB in C', AD in A'', BD in B'', CD in C''; prove that the lines A'A'', B'B'', C'C'' meet in a point which also lies on the curve. [See Quest. 7058 below.] 95

7045. (Dr. Curtis.) — Rays proceeding from any point (or points) on either of the focal conics of a confocal system of surfaces of the second degree are reflected by any of these surfaces; prove that the reflected rays all pass through the same focal conic. 53

7048. (Dr. Curtis.) — An ellipse is placed with its major axis horizontal; prove that the straight line of quickest descent from either focus to the curve (or conversely) is given by the equation $r^2 - 3ar + 2b^2 = 0$, and that *only one* of the roots of this equation for r , the focal radius vector, is admissible. 23

7054. (Professor Malet, M.A.) — If from any point on the outer of two similar and coaxial ellipsoids a tangent cone be drawn to the inner, prove that the volume contained between the surfaces of the cone and either of the ellipsoids is constant. 44

7055. (Professor Cavallin, M.A.)—From an origin, within a closed convex contour, a perpendicular p , making an angle θ with some fixed direction, is drawn on the tangent at P, show that, if ρ be the radius of curvature at P, and ψ some function of θ , then is in general

$$\int_0^{2\pi} \left(p\psi - \frac{dp}{d\theta} \frac{d\psi}{d\theta} \right) d\theta = \int_0^{2\pi} \rho\psi d\theta. \dots\dots\dots 89$$

7058. (J. J. Walker, M.A.)—Show that the above Question 7043 may be generalized thus:—If conics pass through four given points on a cubic, the chords of their remaining points of intersection with it meet in one point, which also lies on the cubic. 95

7061 & 7095. (J. Hammond, M.A.)—Prove that, (1) if we have

$$\phi(x) = \phi\left(\frac{c}{x-1}\right), \quad \text{then} \quad \int_0^x \frac{\phi(x) dx}{x^2-x-c} = \frac{1}{2} \int \frac{\phi(x) dx}{x^2-x-c};$$

and (2), if we have $\phi(x) = \phi\left(\sin^2\alpha + \frac{\cos^2\alpha}{x}\right)$,

$$\text{then} \quad \int_0^\infty \frac{\phi(x) dx}{(x-1)(x+\cos^2\alpha)} = 2 \int_0^1 \frac{\phi(x) dx}{(x-1)(x+\cos^2\alpha)} = 0,$$

$$\text{and also} \quad \int_0^\infty \frac{\phi(x) dx}{(x-1)^2} = (1+\sec^2\alpha) \int_0^1 \frac{\phi(x) dx}{(x-1)^2}. \dots\dots\dots 11$$

7063. (R. F. Scott, M.A.)—A and B are partners in a business in which their interests are in the ratio of a to b . They admit C to the partnership (without altering the capital) in such a way that the interests of the three partners in the business are then equal. C pays £ c for the privilege; how is this to be divided between A and B? 46

7067. (W. S. McCay, M.A.)—Show that the problem, "To construct a triangle of given species and minimum area with its vertices on

the sides of a given triangle (Δ),” admits of two solutions Δ_1 and Δ_2 ; and if the perpendiculars to the sides of Δ at the vertices of Δ_1 and Δ_2 meet in P_1 and P_2 respectively, then P_1, P_2 are inverse points to the circumscribed circle of Δ , and $\Delta_1 : \Delta_2 = OP_1 : OP_2$, where O is the centre of the circle, and Δ_1, Δ_2 are the two minima areas. 91

7068. (W. J. C. Sharp, M.A.)—Show that, if n be any integer, $n^5 - n$ is divisible by 30, and by 240 when n is odd. 45

7069. (D. Edwardes.)—If x, y, z be the distances of a point P from the angular points of a triangle, prove that the mean value of $x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C$, as P ranges over the circle about ABC , is three times the area of the triangle. 65

7073. (Professor Sylvester, F.R.S.)—If from each point of inflexion tangents are drawn to a cubic curve, meeting it in 27 points, and from each of these tangents meeting it in 108 points, and again from each of these tangents meeting it in 432 points, and so on; prove that the $27 \cdot 4^{i-1}$ points so obtained by applying this process i times will be the complete system of points in which proper curves of the order 2^i can be drawn having $3 \cdot 2^i$ consecutive points in common with the cubic. 37

7077. (Professor Minchin, M.A.)—Prove that (1) the total energy of a rigid body is equal to the energy due to the velocity of translation of P + the energy due to the rotation round P , if P is any point on the quadric whose equation referred to the principal axes at G (the centre of mass) is $\lambda^2 + \mu^2 + \nu^2 + a\lambda + b\mu + c\nu = 0$, where (a, b, c) are the velocity components of G , and $\lambda \equiv \omega_1 x - \omega_2 y$ &c.; and (2) in uniplanar motion the locus of P is the circle described on GI as diameter ($I \equiv$ instantaneous centre). [This property has been always enunciated heretofore as if belonging to G exclusively.] 37

7079. (The Editor.)—Prove that, if
 $ax + cy : by + dz = ay + cz : bz + dx = az + cx : bx + dy$,
 then (1) each ratio $= a + c : b + d$, and (2) $\Sigma x^2 = \Sigma (yz)$ 38

7086. (G. F. Walker, M.A.)—If the surface
 $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2Ax + 2By + 2Cz + D = 0$
 represent a paraboloid of revolution, prove that, provided f, g, h be finite, its focus and directrix plane are given by the equations

$$\begin{aligned} f \left[x + \frac{2A}{a+b+c} \right] &= g \left[y + \frac{2B}{a+b+c} \right] \\ &= h \left[z + \frac{2C}{a+b+c} \right] = fgh \frac{2 \frac{A^2+B^2+C^2}{a+b+c} - D}{2 (Ag h + Bh f + Cf g)} \\ 2 (Ag h + Bh f + Cf g) (g h x + h f y + f g z) \\ &+ (A^2 + B^2 + C^2) f g h + D (g^2 h^2 + h^2 f^2 + f^2 g^2) = 0. \dots 85 \end{aligned}$$

7093. (H. L. Orchard, M.A.)—If C, S , respectively, be the centre and one focus of the ellipse $3(x^2 - a^2) + 4y^2 = 0$, and P be a point on the curve such that the $\angle CSP = \theta$; find (1) the area of the triangle CSP , and (2) for what value of θ this will be a maximum. 65

7094. (D. Edwardes.)—Prove that the area of a triangle in terms of the radii (ρ_1, ρ_2, ρ_3) of the escribed circles of its orthocentric triangle is

$$\frac{1}{2} (\rho_1 + \rho_2) (\rho_2 + \rho_3) (\rho_3 + \rho_1) (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1)^{-\frac{1}{2}}. \dots 56$$

7096. (R. Knowles, B.A.)—If m, n are positive integers, prove that

$$\begin{aligned} & \frac{2^n}{m+1} - \frac{n \cdot 2^{n-1}}{(m+1)(m+2)} + \frac{n(n-1)2^{n-2}}{(m+1)(m+2)(m+3)} \cdots \\ & \cdots + (-1)^n \frac{n(n-1)(n-2) \cdots 1}{(m+1)(m+2) \cdots (m+n+1)} \\ & = \frac{1}{m+1} + \frac{n}{m+2} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{m+3} \cdots + \frac{1}{m+n+1} \dots\dots\dots 54 \end{aligned}$$

7104. (Professor Burnside, M.A.)—Show that (1) the general differential equation of lines of the second order may be put under the

$$\text{form } \frac{d^3}{dx^3} \left\{ \left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} \right\} = 0; \quad (2) \quad \frac{d^2}{dx^2} \left\{ \left(\frac{d^2y}{dx^2} \sin \omega \right)^{-\frac{2}{3}} \right\} = -K - \frac{2}{3},$$

where K is that absolute invariant of the conic which depends upon the size only, the area being πK ; (3) hence also the general differential equation of all parabolas may be written

$$\frac{d^2}{dx^2} \left\{ \left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} \right\} = 0. \dots\dots\dots 71$$

7105. (Professor Minchin, M.A.)—If (a, b, s) denote the elongations and shear with reference to any pair of rectangular axes at a point P in a natural solid subject to uniplanar strain, and if k_1, k_2 denote the radii of gyration, about the axes, of any small area, S , surrounding P in the plane of strain; show that the quantity $ak_1^2 + bk_2^2 - 2sp^2$ has the same value for all sets of axes at P , where Sp^2 denotes the product of inertia of S with respect to the axes. 72

7106. (Professor Hudson, M.A.)—Four heavy equal rough particles at the corners of a square are connected by smooth rigid weightless wires and rest on a horizontal table. A force is applied to the system along the wire joining two of the particles, and is increased until the particles are on the point of motion. Obtain an equation to determine the point about which the system begins to turn. 111

7107. (Professor Malet, M.A.)—If the roots $x_1, x_2, x_3, x_4, x_5, x_6$ of the sextic equation $x^6 + p_1x^5 + p_2x^4 - p_3x^3 + p_4x^2 - p_5x + p_6 = 0$ be connected by the relation $x_1x_3x_5 = x_4x_6x_2$, prove that x_1, x_2, x_3 are the roots of the

$$\begin{aligned} & \text{cubic} \quad x^3 - \frac{1}{2} \left\{ p_1 + \left(p_1^2 - 4p_2 + \frac{4p_5}{\sqrt{p_6}} \right) \right\} x^2 \\ & + \frac{1}{2} \left\{ \frac{p_5}{\sqrt{p_6}} + \left(\frac{p_5^2}{p_6} - 4p_4 + 4p_1\sqrt{p_6} \right)^{\frac{1}{2}} \right\} x - \sqrt{p_6} = 0. \dots\dots 76 \end{aligned}$$

7108. (The Editor.)—Into a full conical wine-glass whose depth is a and generating angle α , there is dropped a spherical ball that causes the greatest overflow; show that (1) the radius of the ball is $\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}$,

and (2) when $\alpha = 30^\circ$, the radius of the ball is $\frac{1}{2}a$, and its centre is at the middle of the top of the wine-glass. 48

7109. (J. J. Walker, M.A.)—Show, from its fundamental property, that a principal axis (x) through any origin is determined by $\nabla \cdot a \Sigma m \rho a \rho = 0$, ρ being the vector of any element m of the body; and hence deduce some of the derived properties. 109

a_1, b_1 the semi-axes of the confocal ellipse through (x', y') , show that

$$\sin^2 \alpha = \frac{a_1^2 a^2 y'^2 + b_1^2 b^2 x'^2}{a_1^4 y'^2 + b_1^4 x'^2} \dots\dots\dots 59$$

7169. (Professor Hudson, M.A.)—The sum of the three sides of a right-angled spherical triangle is a quadrant, prove that the least value of the hypotenuse is $\cos^{-1} \frac{1}{2}$, and that in this case the spherical excess is $\sin^{-1} \frac{1}{2}$. $\dots\dots\dots 89$

7176. (R. F. Scott, M.A.)—Through the middle point of the radius vector of the lemniscate $r^2 = a^2 \cos 2\theta$, a perpendicular is drawn to the radius vector; prove that the locus of the point in which this meets the normal to the lemniscate, at the extremity of the radius vector, is the hyperbola $4r^2 \cos 2\theta = a^2$. $\dots\dots\dots 108$

7180. (A. McMurchy, B.A.)—Prove that the whole volume of the solid bounded by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1, \text{ is } \left(\frac{5 \cdot 7}{9 \cdot 11 \cdot 13 \cdot 17 \cdot 19}\right) 4\pi abc. \quad 103$$

7190. (Professor Wolstenholme, M.A.)—If x, y, z be three quantities satisfying the two symmetrical equations

$$yz + zx + xy = 0, \quad x^3 + y^3 + z^3 + 4xyz = 0;$$

prove that (1) they will also satisfy one of the two pairs of semi-symmetrical expressions

$$y^2z + z^2x + x^2y = (y-z)(z-x)(x-y), = +xyz,$$

$$yz^2 + zx^2 + xy^2 = (y-z)(z-x)(x-y), = -xyz;$$

and (2) one set of the following equations will also be satisfied:—

$$(x^2 + yz - y^2 = 0, \quad y^2 + zx - z^2 = 0, \quad z^2 + xy - x^2 = 0);$$

$$(x^2 + yz - z^2 = 0, \quad z^2 + zx - x^2 = 0, \quad x^2 + xy - y^2 = 0). \quad \dots\dots\dots 87$$

7196. (Professor Townsend, F.R.S.)—Perpendiculars being supposed let fall from the centre of an ellipsoid abc upon the several osculating planes to the curve of intersection of any two confocal quadrics $a_1b_1c_1$ and $a_2b_2c_2$; investigate symmetrically the equation of the cone they determine, and show from it that the several sections of the cone by parallels to the principal planes of the system are the evolutes of conics.

$\dots\dots\dots 103$

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6782. (By R. A. ROBERTS, M.A.)—In the motion of a rigid body about a fixed point under the action of no forces, show that the plane, containing two positions of the invariable line in the body which are separated by a constant interval of time, touches a cone concyclic with the invariable cone.

Solution by Professor TOWNSEND, F.R.S.

Assuming that a great circle of a sphere, which cuts off a constant spherical area from any fixed conic on the sphere envelopes a con-cyclic conic on the sphere, in order to prove the above pretty property, it is only necessary to show that, for every two positions OP_1 and OP_2 of the invariable line in the body, its two momentary changes of position P_1OQ_1 and P_2OQ_2 are inversely as the sines of the angles which their planes make with the plane of connection P_1OP_2 of the positions. This may be readily done as follows.

It was shown by MACCULLAGH, at his lectures on the rotation of a rigid body round a fixed point under the circumstances supposed in the Question (see his *Collected Works*, p. 338), that, in any position OP of the invariable line in the body, its momentary change of position POQ varies as the

quantity $\left\{ \left(\frac{a^2 - r^2}{a^2} \right)^2 x^2 + \left(\frac{b^2 - r^2}{b^2} \right)^2 y^2 + \left(\frac{c^2 - r^2}{c^2} \right)^2 z^2 \right\}^{\frac{1}{2}}$,

where a, b, c are the semi-axes of the ellipsoid of inertia of the body with respect to the centre of rotation O , r the constant length of the invariable radius of the motion, and x, y, z the coordinates perpendicular to the principal planes of the surface of the corresponding position P of the invariable point in the body. Hence, in the spherical triangle P_1IP_2 determined, on the sphere of radius r round centre O , by the plane P_1OP_2 connecting any two positions OP_1 and OP_2 of the invariable line, with the two tangent planes OP_1I and OP_2I to the invariable cone corresponding to the positions, it is to be shown that

$$\left\{ \left(\frac{a^2 - r^2}{a^2} \right)^2 x_1^2 + \&c. \right\} \sin IP_1P_2 = \left\{ \left(\frac{a^2 - r^2}{a^2} \right)^2 x_2^2 + \&c. \right\}^{\frac{1}{2}} \sin IP_2P_1,$$

or, if P be the pole on the sphere of the great circle P_1P_2 , that

$$\left\{ \left(\frac{a^2 - r^2}{a^2} \right)^2 x_1^2 + \&c. \right\}^{\frac{1}{2}} \cos IP_1P = \left\{ \left(\frac{a^2 - r^2}{a^2} \right)^2 x_2^2 + \&c. \right\}^{\frac{1}{2}} \cos IP_2P.$$

But, the equation of the invariable cone being $\left(\frac{a^2 - r^2}{a^2} \right) x^2 + \&c. = 0$, and

those of the tangent planes to it along OP_1 and OP_2 being consequently $\left(\frac{x^2 - a^2}{a^2} \right) x_1 \cdot x + \&c. = 0$ and $\left(\frac{x^2 - a^2}{a^2} \right) x_2 \cdot x + \&c. = 0$, it is easily seen,

after a few trifling reductions, that the two magnitudes to be proved equal, multiplied each by the factor $r \cdot \sin P_1OP_2$, have for common value the symmetrical equivalent

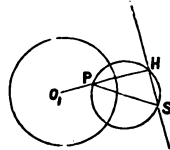
$$\left\{ \left(\frac{a^2 - r^2}{a^2} \right) x_1 x_2 + \left(\frac{b^2 - r^2}{b^2} \right) y_1 y_2 + \left(\frac{c^2 - r^2}{c^2} \right) x_1 x_2 \right\};$$

and therefore $\&c.$, as regards the property in question.

5905. (By Professor COCHERZ.)—Quatre circonférences étant données, trouver sur l'une d'elles un point tel que les polaires de ce point par rapport aux trois autres circonférences le coupent au même point.

Solution by J. L. MCKENZIE, B.A.; E. BUTTER; and others.

Suppose O_1, O_2, O_3 to be the centres of the three circles; P the required point; S the point of intersection of the three polars of P. Let HS be the polar of P with regard to the circle O_1 ; then O_1P is perpendicular to HS; and $O_1H \cdot O_1P = r_1^2$, where r_1 = radius of the circle O_1 . Therefore the circle PHS (on PS as diameter) cuts the circle O_1 orthogonally. Similarly the circle on PS as diameter cuts the circles O_2 and O_3 orthogonally. Hence the construction; draw the circle cutting orthogonally the three circles O_1, O_2, O_3 ; either of the points in which it cuts the fourth given circle may be taken as the required point.



[The problem is impossible in the following cases:—

- (1) If, of the three circles O_1, O_2, O_3 , each cuts the other two, seeing that, then, no circle can be drawn cutting the three orthogonally; (2) if the circle orthogonal to $O_1O_2O_3$ does not meet the fourth circle.]

6905. (By Professor HUDSON, M.A.)—A heavy pulley, of weight R, hangs by a string over a fixed pulley, the other end of the string being attached to a weight W; over the pulley R hangs a string with unequal weights P, Q at its ends; find the accelerations of P, Q, W; and, as an example, let P = 5 lbs., Q = 4 lbs., R = 2 lbs., W = 12 lbs.

Solution by G. HEPPLE, M.A. ; C. BICKERDIKE ; and others.

Let the accelerations of P, Q, R, W, be respectively pg, qg, rg, wg , all being estimated downwards ; then, equating the tensions of the two parts of lower and upper strings, and expressing the condition that the strings may remain stretched, we have

$$P(1-p) = Q(1-q), \quad W(1-w) = R(1-r) + 2P(1-p),$$

$$w+r=0, \quad p-r+q-r=0,$$

4 equations to determine p, q, w, r . Putting $W+R=U, W-R=V, P+Q=S, P-Q=D$, the results are

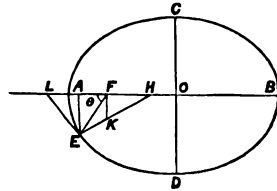
$$w = \frac{VS-4PQ}{US+4PQ}, \quad p = -\frac{VS-4PQ-2DW}{US+4PQ}, \quad q = -\frac{VS-4PQ+2DW}{US+4PQ}.$$

Substituting the numbers given in the question, we get the acceleration of $W = \frac{1}{16\frac{1}{3}}g$ downwards, of $P = \frac{1}{16\frac{1}{3}}g$ downwards, and of $Q = \frac{1}{16\frac{1}{3}}g$ upwards.

7048. (By Dr. CURTIS.)—An ellipse is placed with its major axis horizontal ; prove that the straight line of quickest descent from either focus to the curve (or conversely) is given by the equation $r^2 - 3ar + 2b^2 = 0$, and that *only one* of the roots of this equation for r , the focal radius vector, is admissible. [See Question 6970, *Reprint*, Vol. xxxvii., p. 62.]

Solution by J. YOUNG, M.A. ; ROBERT RAWSON ; and others.

Let the ellipse be represented in the figure, O being its centre and F the focus with which we are concerned. The radius vector FE, which solves the problem, must be such that a circle can be described which shall touch the major axis AB at F and the ellipsoid at E ; the point E must therefore be such that, drawing the normal EH cutting in K the perpendicular to AB at F, FK shall = EK, and therefore, drawing the tangent EL, EL = LF. Denoting, then, the angle AFE by θ , we have $2LF \cos \theta = r$; or, if x be the abscissa of E referred to the axes of the ellipse,



$$2 \left(\frac{a^2}{x} - ae \right) \cos \theta = r, \quad \text{or} \quad 2a(a-ex) \cos \theta = rx ;$$

but $r = a-ex$, therefore $2a \cos \theta = x$, or $2ae \cos \theta = ex = a-r$;

but, by the equation of the ellipse, $r = \frac{b^2}{1+e \cos \theta}$, $\therefore 2a \left(\frac{b^2}{ar} - 1 \right) = a-r$,

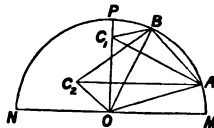
or $r^2 - 3ar + 2b^2 = 0$; hence $r = \frac{3a}{2} \pm \frac{a}{2} \left(9 - 8 \frac{b^2}{a^2} \right)^{\frac{1}{2}}$; but, as $9 - 8 \frac{b^2}{a^2} > 1$,

and $r < 2a$, the upper sign is excluded.

6129. (By Professor MATZ, M.A.)—Two points are taken at random in the arc of a given circular quadrant of radius 1; and a third point is taken at random, (1) in one of the rectilinear sides of the quadrant, and (2) in the surface of the adjacent quadrant; and these are joined, in their respective order, by straight lines; prove that the mean areas of the triangles thus formed are $\Delta_1 = \frac{r^2}{\pi^2} (3\pi - 8)$, $\Delta_2 = \frac{r^2}{\pi^2} (2\pi - 4)$.

Solution by Professor SEITZ, M.A.; D. EDWARDS; and others.

Let MOP and NOP be the two quadrants;
A, B two random points in the arc MP; C_1, C_2
random points in OP and the surface NOP,
respectively; $OM = r$, $OC_1 = x$, $OC_2 = y$,
 $\angle AOP = \theta$, $\angle BOP = \phi$, $\angle POC_2 = \psi$, area
 $ABC_1 = u_1$, and area $ABC_2 = u_2$; then



$$u_1 = \frac{1}{2}r^2 \sin(\theta - \phi) + \frac{1}{2}rx \sin \phi - \frac{1}{2}rx \sin \theta,$$

$$u_2 = \frac{1}{2}r^2 \sin(\theta - \phi) + \frac{1}{2}ry \sin(\phi + \psi) - \frac{1}{2}ry \sin(\theta + \psi); \text{ hence}$$

$$\Delta_1 = \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_0^r u_1 r d\theta r d\phi dx + \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_0^r r d\theta r d\phi dx$$

$$= \frac{2r^2}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \{2 \sin(\theta - \phi) + \sin \phi - \sin \theta\} d\theta d\phi$$

$$= \frac{2r^2}{\pi^2} \int_0^{\frac{1}{2}\pi} (3 - 3 \cos \theta - \theta \sin \theta) d\theta = \frac{r^2}{\pi^2} (3\pi - 8);$$

$$\Delta_2 = \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_0^r u_2 r d\theta r d\phi dy + \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_0^r r d\theta r d\phi d\psi y dy$$

$$= \frac{8r^2}{3\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_0^{\frac{1}{2}\pi} \{3 \sin(\theta - \phi) + 2 \sin(\phi + \psi) - 2 \sin(\theta + \psi)\} d\theta d\phi d\psi$$

$$= \frac{4r^2}{3\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \{3\pi \sin(\theta - \phi) - 4 \sin \theta - 4 \cos \theta + 4 \sin \phi + 4 \cos \phi\} d\theta d\phi$$

$$= \frac{4r^2}{3\pi^3} \int_0^{\frac{1}{2}\pi} \{3\pi - 3\pi \cos \theta - 4\theta \sin \theta - 4\theta \cos \theta + 4 + 4 \sin \theta - 4 \cos \theta\} d\theta$$

$$= \frac{r^2}{\pi^2} (2\pi - 4).$$

6805. (By A. McINTOSH, M.A.)—In Question 6784, prove that (1) the line joining the points of contact of the conic in (14) with the nodal tangents, passes through the points of inflexion of the cubic; (2) if two imaginary tangents be drawn from the point R (i.e., the third point in which TT' meets the curve), prove that the chord joining the imaginary points of contact will pass through the primary point P; (3) the point in which PN meets TT' lies on another nodal cubic which has the same

nodal tangents and the same conic as in (14, 6487); (4) if two real tangents be drawn from the point of intersection of PN and TT' to this latter cubic, the line joining the points of contact is the same as the line mentioned in (2), and therefore passes through P; (5) a line drawn through a fixed point on a nodal cubic intersects the curve again in two points; if two pairs of tangents be drawn from these points to the curve, the chords joining their points of contact intersect on a fixed line passing through the node; (6) the point in which this fixed line meets the curve, and the line joining the first fixed points to the node, bear the same relation to one another; (7) in a cuspidal cubic, if the line joining any two points on the curve subtends a right angle at the cusp, it touches a conic which passes through the cusp, and cuts the cubic there at right angles.

Solution by the PROPOSER; G. F. WALKER, M.A.; and others.

Equation of Folium being $x^3 + y^3 = 3axy$, any point on the curve may

be given by the equations $x = \frac{3a\mu}{1+\mu^3}$ $y = \frac{3a\mu^2}{1+\mu^3}$,

μ being the tangent of the angle that the line joining the point to the node makes with one of the nodal tangents, if the nodal tangents are at right angles.

Let a tangent touch at (xy) and pass through $(x'y')$, the latter being a point on the curve. Then

$$(x^2 - ay)(x - x') + (y^2 - ax)(y - y') = 0.$$

This reduces to

$$(x^2 - ay)x' + (y^2 - ax)y' = axy,$$

by the equation of the curve.

Substitute in this equation the values given above for xy , and for $x'y'$

substitute respectively $\frac{3a\mu'}{1+\mu'^3}$ and $\frac{3a\mu'^2}{1+\mu'^3}$. It will be found that the equation may be reduced to $(\mu - \mu')^2(\mu^2\mu' + 1) = 0$. This shows that $\mu^2\mu' = -1$. If μ' be positive, *i.e.*, if the point with which we start lie on the loop, the values of μ are imaginary, therefore two real tangents cannot be drawn from any point on the loop.

The two values of μ are equal and opposite in sign, therefore NT + NT' make equal angles with the nodal tangents in the Folium. The coordinates of the middle point of TT' are

$$\frac{1}{2} \left(\frac{3a\mu}{1+\mu^3} - \frac{3a\mu}{1-\mu^3} \right) \text{ and } \frac{1}{2} \left(\frac{3a\mu^2}{1+\mu^3} + \frac{3a\mu^2}{1-\mu^3} \right).$$

These expressions become $-\frac{3a\mu^4}{1-\mu^6}$ and $\frac{3a\mu^2}{1-\mu^6}$ respectively. But the co-

ordinates of P are $\frac{3a\mu'}{1+\mu'^3}$ and $\frac{3a\mu'^2}{1+\mu'^3}$, *i.e.*, $\frac{3a\mu^4}{1-\mu^6}$ and $-\frac{3a\mu^2}{1-\mu^6}$.

These are equal and opposite in sign to the coordinates of M, the middle point of TT'; therefore PM passes through N, and is bisected in N.

The equation of the line TT' is easily found. It is

$$\frac{x - \frac{3a\mu}{1+\mu^3}}{\frac{3a\mu}{1+\mu^3} + \frac{3a\mu}{1-\mu^3}} = \frac{y - \frac{3a\mu^2}{1+\mu^3}}{\frac{3a\mu^2}{1+\mu^3} - \frac{3a\mu^2}{1-\mu^3}},$$

$$\text{therefore } \frac{x(1+\mu^3-3\mu)}{6\mu} = \frac{y(1+\mu^3)-3\mu^2}{-6\mu^3}.$$

$$\text{Hence } (x\mu^4+y)(1+\mu^3)-3\mu^2(1+\mu^3)=0, \text{ or } x\mu^4+y-3\mu^2=0.$$

If a tangent be drawn at the point P, the μ of the point in which it meets the curve again is $-\frac{1}{\mu^2}$; but $\mu' = -\frac{1}{\mu^2}$, therefore the μ of the point in which it meets the curve again is $-\mu^4$. Hence the equation of the line joining N to this point is $y+x\mu^4=0$. This line is consequently parallel to TT', and therefore the line PQ in the question is bisected by the curve.

It is evident, from the equation of the line TT', that it envelopes the rectangular hyperbola, $4xy=9a^2$. The hyperbola touches the curve in the point $(\frac{3a}{2}, \frac{3a}{2})$, and this point is the vertex. It will be easily seen

that the chords joining real points of contact touch one branch of the hyperbola, and the chords joining imaginary points of contact, the other.

If the tangent PQ cut the curve in S, it follows from the general theory of cubics that RS is also a tangent to the cubic at the point R. The chord RS, therefore, bears the same relation to the cubic as TT', and therefore also touches the rectangular hyperbola mentioned above. The lines NT and NT' make equal angles with a nodal tangent and for the same reason, the lines NM and NR make equal angles with it, therefore $\angle TNM = \angle TNR$.

The middle point M of TT' lies on an identical Folium, because MN is equal to PN and is in an opposite direction. It is thus easy to see that the equation of this Folium is $x^3+y^3=-3axy$, and that its loop lies in the opposite quadrant.

The coordinates of the point M are $-\frac{3a\mu'}{1+\mu'^3}$ and $-\frac{3a\mu'^2}{1+\mu'^3}$: the equation of the chord joining the points of contact of tangents from this point to the curve $x^3+y^3=-3axy$ is therefore

$$\frac{1}{\mu'^3}x+y-\frac{3a}{\mu'}=0, \text{ i.e., } x+\mu'^2y-3a\mu'=0.$$

This line passes through the point $(\frac{3a\mu'}{1+\mu'^3}, \frac{3a\mu'^2}{1+\mu'^3})$, as may be verified by

substitution, and it also touches one branch of the rectangular hyperbola mentioned before. It also passes through the imaginary points of contact of tangents from the point R to the original curve. This is very much like the theory of poles and polars with respect to curves of the second degree.

If T'N be produced on to meet the tangent PT, we see, by the theory of transversals, that

$$TT' \cdot MN \cdot PU = MT' \cdot PN \cdot TU;$$

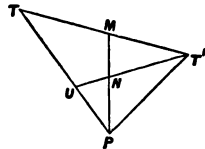
but $TT' = 2MT'$ and $MN = PN$,

therefore $TU = 2PU$;

similarly $TM \cdot T'N \cdot TP = T'M \cdot NU \cdot PU$;

but $TM = T'M$ and $TP = 3PU$

therefore $T'N = 3NU$.



Therefore U, the point of trisection of PT most remote from the point of contact, lies on a Folium, the linear dimensions of which are one-third those of the original. The centroid of the triangle PTT' evidently lies on this Folium. It is a point of trisection of the tangent at R to the curve. This hardly requires a proof.

Taking the general nodal cubic $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 3exy$, and proceeding in the same manner, we may prove that $\mu^2\mu' = -\frac{a}{d}$. How-

ever, it will be simpler to find the μ 's for the points in which any line $y = mx + n$ cuts this curve. The μ 's will be the three ratios of y to x given by the cubic equation $(ax^3 + 3bx^2y + 3cxy^2 + dy^3)n = 3exy(y - mx)$.

Hence $\mu\mu'\mu'' = -\frac{an}{dn} = -\frac{a}{d}$. If the line be a tangent, two of the μ 's

will be equal, therefore $\mu^2\mu' = -\frac{a}{d}$; $\mu = \pm \sqrt{-\frac{a}{d\mu'}}$, $\therefore$ NT and NT' form

a harmonic pencil with the nodal tangents. A point on the curve is now

given by $x = \frac{3e\mu}{a + 3b\mu + 3c\mu^2 + d\mu^3}$, $y = \frac{3e\mu^2}{a + 3b\mu + 3c\mu^2 + d\mu^3}$.

Let this be the point T. The coordinates of the point T' are

$$\frac{-3eu}{a - 3b\mu + 3c\mu^2 - d\mu^3} \text{ and } \frac{3e\mu^2}{a - 3b\mu + 3c\mu^2 - d\mu^3}.$$

The equation of the chord TT' can now be worked out as before. It will be found to be divisible by $a + 3b\mu + 3c\mu^2 + d\mu^3$ finally, just as the other was divisible by $1 + \mu^3$. It becomes $\mu^4dx + (3bx + 3cy - e)\mu^2 + ay = 0$, and therefore touches the conic $4adxy = 9(bx + cy - e)^2$, a conic which evidently touches the nodal tangents, the chord of contact being

$$bx + cy - e = 0.$$

6744. (By Dr. MACFARLANE, F.R.S.E.)—Deduce the necessary consequences of the law that a man cannot be the husband of his wife's sister.

Solution by the PROPOSER.

Let A denote any subject of a nation in which the above law is established; then mA will denote any male subject, and $mc^{-1}cfc^{-1}fc^{-1}cmA$ will denote any husband of any sister of any wife of the man A. Hence $\Sigma mc^{-1}cfc^{-1}fc^{-1}cmA$ will denote all the husbands of all the sisters of all the wives of the man A. The law referred to provides that A and any member of the above class cannot be identical, which is expressed by

$$\Sigma A . mc^{-1}cfc^{-1}fc^{-1}cmA = 0 \dots \dots \dots (1),$$

where the left-hand side denotes all the members of the compound class formed by taking simultaneously the two classes A and $\Sigma mc^{-1}cfc^{-1}fc^{-1}cmA$. The rule for transforming such a universal equation is as follows:—Suppose all the relationship symbols brought over to one factor of the com-

pound class in accordance with the rule given in solution to Question 6677 (*Reprint*, Vol. XXXV, p. 110); then, removing a symbol from the front gives one derived equation, and removing a symbol from the end gives another derived equation; transform each of these derived equations in a similar manner, then each of their four resultants, and so on, until all the relationship symbols have been brought to the other side. The total of these derived equations forms the total of the transformations of the given universal equation. The first derived equation in the above case is

$$\Sigma c m A . c f c c^{-1} f c^{-1} c m A = 0 \dots\dots\dots(2);$$

that is, a child of a man cannot be the child of a sister of a wife of the man. The second derived equation is

$$\Sigma m c^{-1} A . m c^{-1} c f c c^{-1} f c^{-1} A = 0 \dots\dots\dots(3);$$

that is, the father of a person cannot be the husband of a sister of the mother of the person. By applying the process to (2) and (3), and so on, other 19 different forms are obtained, which respectively express that—

- A wife of a man cannot be the sister of a wife of the man.
- A child of the father of a person*,, the child of a sister of the mother of the person.
- A husband of a woman,, the husband of a sister of the woman.
- A parent of a wife of a man,, the parent of another wife of the man.
- A wife of the father of a person,, the sister of the mother of the person.
- A child of a husband of a woman,, the child of a sister of the woman.
- A husband of a daughter of a person,, the husband of another daughter of the person.
- A parent of a step-mother of a person,, the parent of the mother of the person.
- A wife of a husband of a woman,, the sister of the woman.
- A child of a husband of a daughter of a person,, the child of another daughter of the person.
- A sister of the wife of the father of a person,, the mother of the person.
- A parent of another wife of a husband of a woman,, the parent of the woman.
- A second wife of a husband of a daughter of a person,, the daughter of the person.
- A child of a husband of a sister of a woman,, the child of the woman.
- A child of a sister of a wife of the father of a person,, the person.
- A sister of a wife of a husband of a woman,, the woman.
- A parent of another wife of a husband of a daughter of a person,, the person.
- A wife of a husband of a sister of a woman,, the woman.
- A child of a husband of a sister of the mother of a person,, the person.

6913, 6940. (By Rev. A. J. C. ALLEN, M.A., and J. YOUNG, M.A.)—
If K be the orthocentre of a triangle, P any point on the circumscribing circle, and if PK cut the pedal line of P in Q; prove that (1) PK is bisected in Q, (2) the locus of Q is the nine-point circle.

* The symbol ,, is an ellipse for *cannot lawfully be*.

6916. (By R. TUCKER, M.A.)—ABC is a triangle, and PQR is an equilateral triangle in the circle ABC; prove that (1) the pedal lines, corresponding to the points P, Q, R, form another equilateral triangle; (2) the pedal lines, corresponding to points at equal arcual distances from any angle (A), intersect on a straight line.

Solution by CHARLOTTE A. SCOTT; SARAH MARKS;
T. WOODCOCK, B.A.; and others.

(6913, 6940). Since (*Reprint*, XXXIV, 25) KP is bisected in Q, if we join K to O, the centre of the circumscribing circle, and bisect KO in S, we have

$$SQ = \frac{1}{2}OP;$$

hence the locus of Q is the nine-point circle.

(6916). 1. The angle $PMA = HVM$,
and the angle $QMA = H'VM$;
thus, if HP' and $H'Q'$ intersect in S, so that

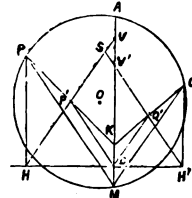
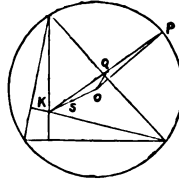
$$\angle HSH' = HVM + H'VM,$$

we have $\angle HSH' = PMA + QMA = PMQ = PRQ$;

hence the pedal lines form a triangle similar to the triangle PQR, that is, an equilateral triangle.

2. P is the middle point of KP, and D of KM, hence P'D is parallel to PM, that is, P'D and P'V are equally inclined to VD. Similarly; Q'D and Q'V' are equally inclined to VD. If $AP = AO$, $\angle AMP = \angle AMQ$, and therefore $\angle P'DV = \angle Q'DV' = \angle Q'V'D$, therefore DP'SQ' is a parallelogram, therefore DS and P'Q' bisect each other. But P'Q' is fixed in direction, and is a chord of the nine-point circle, therefore the middle point of P'Q' lies on a fixed diameter of the nine-point circle, that is, the middle point of DS lies on a fixed line, and therefore, D being a fixed point, the locus of S is a straight line.

[Any point P on the circumscribed circle will be the focus of a parabola touching the sides of the triangle; hence the feet of the perpendiculars from P upon the sides of the triangle lie on a straight line, viz., the tangent at the vertex of this parabola; whence we have a proof of the existence of the pedal line. The directrix of this parabola passes through the orthocentre, and is parallel to the pedal line of P. Therefore the straight line from P to the orthocentre is bisected by the pedal line, which proves the property in Question 6940. The centroid and the orthocentre of a triangle are the (internal and external) centres of similitude of the nine-point and circumscribing circles, whose radii are in the ratio 1 : 2; hence, if K be the orthocentre, the mid-point of PK lies on the nine-point circle, which proves Question 6913.]



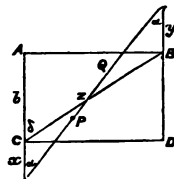
6135. (By W. S. B. WOOLHOUSE, F.R.A.S.)—Find (1) the average area of triangles drawn on the surface of a given rectangle, and having

one of the three sides parallel to a given line; and (2) the directions of the given line when the average area is a maximum and a minimum.

Solution by the PROPOSER.

Parallels to the given line will meet only one of the pairs of opposite sides of the rectangle. Let AB and CD be these sides, and the notation as in the diagram; then we have two cases to consider.

1. When PQ crosses AB and CD (see Diagram 1). PQ being the side of the triangle, let p denote the perpendicular from a variable point R as vertex, and a, b the sides AB, AC; then we have



$$\begin{aligned}\Sigma p &= \frac{(x+b)^2 \tan \alpha}{2} \cdot \frac{(x+b) \sin \alpha}{3} - \frac{x^2 \tan \alpha}{2} \cdot \frac{x \sin \alpha}{3} \\ &\quad + \frac{(b+y)^2 \tan \alpha}{2} \cdot \frac{(b+y) \sin \alpha}{3} - \frac{y^2 \tan \alpha}{3} \cdot \frac{y \sin \alpha}{3} \\ &= \frac{\sin^2 \alpha}{6 \cos \alpha} \left\{ (b+x)^3 - x^3 + (b+y)^3 - y^3 \right\}.\end{aligned}$$

Also, when the points P, Q vary in the line,

$$\Sigma PQ = \int x dx \left(\frac{b}{\cos \alpha} - x \right) = \frac{b}{\cos \alpha} \cdot \frac{x^2}{2} - \frac{x^3}{3} = \frac{b^3}{6 \cos^3 \alpha}.$$

Therefore, as regards the line PQ,

$$\Sigma \Delta' = \frac{1}{2} \Sigma p \cdot \Sigma PQ = \frac{b^3 \sin^2 \alpha}{72 \cos^4 \alpha} \left\{ (b+x)^3 - x^3 + (b+y)^3 - y^3 \right\}.$$

To involve variable positions of the line, multiply by $dx \sin \alpha = -dy \sin \alpha$ and integrate, and we obtain

$$\frac{b^3 \sin^3 \alpha}{288 \cos^4 \alpha} \left\{ (b+x)^4 - x^4 - (b+y)^4 + y^4 \right\}.$$

This is to be taken between the limits

$$\left(\begin{matrix} x = 0 \\ y = \frac{a}{\tan \alpha} - b \end{matrix} \right) \text{ and } \left(\begin{matrix} x = \frac{a}{\tan \alpha} - b \\ y = 0 \end{matrix} \right)$$

and the result is

$$\frac{b^4}{72 \cos \alpha} (2a^3 - 3a^2b \tan \alpha + 2ab^2 \tan^2 \alpha - b^3 \tan^3 \alpha) \dots\dots\dots(1).$$

2. When the line PQ crosses AB and AC.

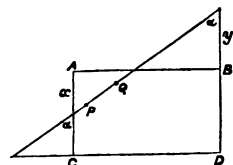
$$\Sigma p = \frac{\sin^2 \alpha}{6 \cos \alpha} \left\{ x^3 + (b+y)^3 - (b-x)^3 - y^3 \right\},$$

$$\Sigma PQ = \frac{x^3}{6 \cos^3 \alpha},$$

$$\Sigma \Delta' = \frac{x^3 \sin^2 \alpha}{72 \cos^4 \alpha} \left\{ x^3 + (b+y)^3 - (b-x)^3 - y^3 \right\}$$

$$= \frac{x^3 \sin^2 \alpha}{72 \cos^4 \alpha} \left\{ x^3 + \left(b + \frac{a}{\tan \alpha} - x \right)^3 - (b-x)^3 - \left(\frac{a}{\tan \alpha} - x \right)^3 \right\}$$

$$= \frac{x^3 \sin^2 \alpha}{72 \cos^4 \alpha} \left\{ \frac{3ab}{\tan \alpha} \left(b + \frac{a}{\tan \alpha} \right) - \frac{6ab}{\tan \alpha} x + 2x^3 \right\},$$



which multiplied by $dx \sin \alpha$ and integrated between the limits $s = b \dots 0$

we obtain
$$\frac{b^4}{72 \cos \alpha} \left(\frac{2}{3} a^2 b \tan \alpha - \frac{2}{3} a b^2 \tan^2 \alpha + \frac{2}{3} b^3 \tan^3 \alpha \right) \dots \dots \dots (2).$$

The same values again recur when the line PQ crosses CD and BD. Hence taking the former result (1), and adding the double of (2), the total sum of all the triangles is found to be

$$\frac{b^4}{72 \cos \alpha} \left(2a^3 - \frac{2}{3} a^2 b \tan \alpha + \frac{4}{3} a b^2 \tan^2 \alpha - \frac{2}{3} b^3 \tan^3 \alpha \right) \dots \dots \dots (3).$$

To ascertain the number of triangles, the positions of P, Q are in the first case
$$\int dx \sin \alpha \cdot \frac{b^2}{2 \cos^2 \alpha} = \frac{b^2}{2 \cos \alpha} (a - b \tan \alpha),$$

and in the second
$$\int dx \sin \alpha \cdot \frac{x^2}{2 \cos^2 \alpha} = \frac{b^2}{2 \cos \alpha} \frac{b \tan \alpha}{3},$$

so that the total positions of P, Q are $\frac{b^2}{2 \cos \alpha} (a - \frac{1}{3} b \tan \alpha)$. And, as each of these has ab positions of the third point R, the total number of triangles is $\frac{ab^3}{2 \cos \alpha} (a - \frac{1}{3} b \tan \alpha)$. Divide (3) by this, and the average triangle is

$$u = \frac{ab}{36} \cdot \frac{2a^3 - \frac{2}{3} a^2 b \tan \alpha + \frac{4}{3} a b^2 \tan^2 \alpha - \frac{2}{3} b^3 \tan^3 \alpha}{a^2 (a - \frac{1}{3} b \tan \alpha)}.$$

To simplify, let $k = \frac{b \tan \alpha}{a} \left(= \frac{\tan \alpha}{\tan \delta} \right);$

then $u = \frac{ab}{36} \cdot \frac{2 - \frac{2}{3} k + \frac{4}{3} k^2 - \frac{2}{3} k^3}{1 - \frac{1}{3} k} = \frac{ab}{840} \left\{ 30k^2 + 13k + 144 - \frac{292}{3-k} \right\}.$

Also $\frac{du}{dk} = \frac{ab}{840} \left\{ 60k + 13 - \frac{292}{(3-k)^2} \right\}, \quad \frac{d^2u}{dk^2} = \frac{ab}{210} \left\{ 15 - \frac{146}{(3-k)^3} \right\}.$

For maxima and minima areas, $\frac{du}{dk}$ put = 0 resolves into

$$(k-1)(60k^2 - 287k + 175) = 0.$$

The root $k = 1$ determines a maximum area, and the inferior root of the quadratic a minimum area. As k must not exceed 1, the superior root is not admissible. The greatest area $(= \frac{1}{18} ab)$ obtains when $k = 0$.

[Another solution has been given on p. 109 of Vol. 37 of the *Reprints*.]

6733. (By G. F. WALKER, M.A.)—Prove that the result of eliminating the arbitrary function from $F(X_1, X_2, \dots, X_r, \dots, X_n) = 0$, where

$$X_r = S^2(x_r - u), \quad S = x_1 + \dots + x_n + u, \quad \text{and } x_1 \dots x_n,$$

are independent variables, is

$$\sum_{r=1}^{r=n} [-\lambda(n+1)x_r + (1+\lambda)S] \frac{du}{dx_r} = -\lambda(n+1)u + (1+\lambda)S.$$

Solution by the PROPOSER; Prof. EVANS, M.A.; and others.

$F = 0$ is the integral of a partial differential equation of the first order, the integrals of the auxiliary equations of which are $X_1 = a_1, \dots, X_n = a_n$, and therefore, from the auxiliary equations, we have $dX_1 = 0 \dots dX_n = 0$,

$$\begin{aligned} \text{therefore} \quad & -\frac{\lambda ds}{s} = \frac{dx_1 - du}{x_1 - u} = \dots = \frac{dx_r - du}{x_r - du} = \dots \\ & = \frac{ds - (x+1) du}{s - (x+1) u} = \frac{-\lambda(n+1) du}{(1+\lambda)s - \lambda(n+1)u} = \frac{-\lambda(n+1) dx_r}{(1+\lambda)s - \lambda(n+1)x_r} = \dots, \\ \text{or} \quad & \frac{dx_1}{-\lambda(n+1)x_1 + (1+\lambda)s} = \dots = \frac{dx_r}{-\lambda(n+1)x_r + (1+\lambda)s} = \dots \\ & \dots = \frac{du}{-\lambda(1+x)u + (1+\lambda)s}. \end{aligned}$$

The differential equation therefore is

$$\sum_{r=1}^{r=n} [-\lambda(n+1)x_r + (1+\lambda)s] \frac{dx_r}{dx_r} = -\lambda(n+1)u + (1+\lambda)s.$$

7015. (By W. J. C. SHARP, M.A.)—Show that (1) if the two conics $S = 0$ and $S' = 0$ have double contact, $F \equiv \lambda S + \mu S'$, $\Phi \equiv \Lambda \Sigma + M \Sigma'$; if λ, μ have the same values as in (1), (2) the equations to the common tangents and common chord are

$$\lambda S - \mu S' = 0 \text{ and } \{(\mu \Delta)^{\frac{1}{2}} S - (\lambda \Delta)^{\frac{1}{2}} S'\}^2 = 0;$$

(3) $\Lambda \Sigma - M \Sigma' = 0$ represents the points of contact; (4) $(\Delta)^{\frac{1}{2}} \Delta' \Sigma - (M)^{\frac{1}{2}} \Delta \Sigma' = 0$ represents the intersection of the common tangents; also (5) furnish an answer to the query, in regard to Question 6204, on p. 75 of Vol. xxxiv. of the *Reprints*.

Solution by W. J. C. SHARP, M.A.

1, 2. If the conics be referred to their common self-conjugate triangle, so that $S \equiv ax^2 + by^2 + cz^2$ and $S' \equiv a'x^2 + b'y^2 + c'z^2$; then for double contact it is necessary and sufficient that two of the fractions $\frac{a'}{a}, \frac{b'}{b}, \frac{c'}{c}$ should be equal. Assume $\frac{a'}{a} = \frac{b'}{b} = r$, then the covariant

$$\begin{aligned} F &\equiv aa'(bc' + b'c)x^2 + bb'(ac' + a'c)y^2 + cc'(ab' + ab)z^2 \\ &\equiv \frac{1}{r} a'b'c'(ax^2 + by^2 + cz^2) + rabc(a'x^2 + b'y^2 + c'z^2) \\ &\equiv \lambda S + \mu S' \text{ if } \lambda = \frac{a'b'c'}{r}, \mu = rabc, \end{aligned}$$

$$\begin{aligned} \text{therefore } \lambda S - \mu S' &\equiv rabc'(ax^2 + by^2 + cz^2) - rabc(rax^2 + rby^2 + cz^2) \\ &\equiv rab(c - rc)(ax^2 + by^2), \end{aligned}$$

$$\begin{aligned} \text{and } (\mu\Delta')^{\frac{1}{2}}S - (\lambda\Delta)^{\frac{1}{2}}S' &\equiv (r\Delta\Delta')^{\frac{1}{2}}S - \left(\frac{\Delta\Delta'}{r}\right)^{\frac{1}{2}}S' \\ &\equiv \left(\frac{\Delta\Delta'}{r}\right)^{\frac{1}{2}}(rS - S') = \left(\frac{\Delta\Delta'}{r}\right)^{\frac{1}{2}}(rc - c')z^2, \end{aligned}$$

which prove the equations to the common tangents and chord.

3, 4. The other equation $\Phi \equiv \Delta\mathfrak{X} + M\mathfrak{X}'$ may be similarly deduced from the tangential equations to the conics, and this leads directly to the conditions given by Mr. WALKER (Question 6204, *Reprint*, Vol. xxxiv., p. 75).

It is easy to show that the ratio $\lambda : \mu$ satisfies the equations

$$\Theta = \frac{\lambda}{\mu} + 2 \left(\frac{\mu\Delta'}{\lambda\Delta} \right)^{\frac{1}{2}}, \quad \Theta' = \frac{\mu\Delta'}{\lambda\Delta} + 2 \left(\frac{\lambda\Delta'}{\mu\Delta} \right)^{\frac{1}{2}}.$$

5. The query, propounded by Professor WOLSTENHOLME in his solution to Question 6204, why cannot the conditions that two conics should have double contact be expressed in terms of their invariants, may be answered in the following manner, which shows that this is a consequence of the fact that it is not possible to express the conditions that a binary quartic should have two pairs of equal roots in terms of its invariants.

If $S=0$ and $S'=0$ be two conics having double contact, the quartic obtained by eliminating one of the coordinates will have two pairs of equal roots, and will differ from its Hessian by a multiplier only. But, since every covariant of a binary quartic can be expressed in terms of the quartic and its Hessian, every such covariant is in this case a multiple of the quartic. If now the coordinate already eliminated between S and S' , be eliminated between $F=0$ (the covariant conic) and $S=0$, the eliminant will be a covariant of the binary quartic, and therefore a multiple of the eliminant of $S=0$ and $S'=0$, and therefore F must be of the form $\lambda S + \mu S'$, the necessary and sufficient condition for double contact.

6957. (By Professor WOLSTENHOLME, M.A.)—Prove that

$$\int_0^{\pi} \frac{\sin^2 nx}{\sin^2(1-n)x} dx = n \int_0^{\pi} \frac{\sin^2 x}{\sin^2(1-n)x} dx = \frac{n}{1-n} \int_0^{\pi} \frac{\sin(1+n)x}{\sin(1-n)x} dx \dots (1),$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 nr}{\sin^2(1-n)r} dr = n \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^2(1-n)x} dx = \frac{n}{1-n} \int_0^{\frac{\pi}{2}} \frac{\sin(1+n)x}{\sin(1-n)x} dx \dots (2),$$

if, in (1), $n > 0 < 2$, and in (2) $n > \frac{1}{2} < \infty$.

Solution by J. HAMMOND, M.A.; CHRISTINE LADD, B.A.; and others.

Only one of each set of these pretty equations is independent, the other being easily deducible from the relation $\sin^2 \alpha - \sin^2 \beta = \sin(\alpha + \beta) \sin(\alpha - \beta)$. Integrating by parts, we have

$$\int \sin^2 \theta \operatorname{cosec}^2(1-n)x \cdot dx = -\frac{\sin^2 \theta}{1-n} \cot(1-n)x + \int \frac{\cot(1-n)x}{1-n} \sin 2\theta \cdot d\theta,$$

and if in this, we take $\theta = x$, and the limits from 0 to π , so that

$$\int_0^\pi \frac{\sin^2 x}{\sin^2(1-n)x} dx = \frac{1}{1-n} \int_0^\pi \sin 2x \cot(1-n)x \cdot dx;$$

then we obtain
$$\int_0^\pi \frac{\sin^2 x}{\sin^2(1-n)x} dx - \frac{1}{1-n} \int_0^\pi \frac{\sin(1+n)x}{\sin(1-n)x} dx$$

$$= \frac{1}{1-n} \int_0^\pi \frac{\sin 2x \cos(1-n)x - \sin[2-(1-n)]x}{\sin(1-n)x} dx = \frac{1}{1-n} \int_0^\pi \cos 2x \cdot dx = 0.$$

Again, in the same, take $\theta = nx$, and the limits from 0 to $\frac{\pi}{n}$, so that

$$\int_0^{\frac{\pi}{n}} \frac{\sin^2 nx}{\sin^2(1-n)x} dx = \frac{n}{1-n} \int_0^{\frac{\pi}{n}} \sin 2nx \cot(1-n)x \cdot dx;$$

then we obtain
$$\int_0^{\frac{\pi}{n}} \frac{\sin^2 nx}{\sin^2(1-n)x} dx - \frac{n}{1-n} \int_0^{\frac{\pi}{n}} \frac{\sin(1+n)x}{\sin(1-n)x} dx$$

$$= \frac{n}{1-n} \int_0^{\frac{\pi}{n}} \frac{\sin 2nx \cos(1-n)x - \sin(2n+1-n)x}{\sin(1-n)x} dx$$

$$= -\frac{n}{1-n} \int_0^{\frac{\pi}{n}} \cos 2nx \cdot dx = 0 \dots \dots \dots (3).$$

As mentioned above, the other results of the question follow at once from (2) and (3); the limitation of the value of n is only introduced in order that the functions under the integral signs may remain finite for all values of x between the limits.

[If AP, BP be straight lines revolving about A, B, with angular velocities in the ratio $1:n$, where n is a proper fraction, the equation of the locus of P is

$$r = a \frac{\sin n\theta}{\sin(1-n)\theta} \quad \text{or} \quad r = a \frac{\sin \frac{\theta}{n}}{\sin\left(\frac{\theta}{n} - \theta\right)},$$

according as A or B is taken for origin; and the area of the outer

$$a^2 \int_0^\pi \frac{\sin^2 n\theta}{\sin^2(1-n)\theta} d\theta \quad \text{or} \quad a^2 \int_0^{n\pi} \frac{\sin^2 \frac{\theta}{n}}{\sin^2\left(\frac{\theta}{n} - \theta\right)} d\theta,$$

which latter
$$= na^2 \int_0^\pi \frac{\sin^2 \theta}{\sin^2(1-n)\theta} d\theta \quad \text{using } n\theta$$

So also the sum of the areas of the two lo

$$a^2 \int_0^{\frac{\pi}{n}} \frac{\sin^2 n\theta}{\sin^2(1-n)\theta} d\theta \quad \text{and} \quad a^2 \int_0^\pi \frac{\sin^2 \theta}{\sin^2(1-n)\theta} d\theta$$

which latter
$$= ra^2 \int_0^{\pi} \frac{\sin^2 \theta d\theta}{\sin^2(1-n)\theta}.$$

The true limits for n , in order that the subjects of integration may remain finite between the limits of integration, are $n > \frac{1}{2} < 2$.

In the same way, it may be proved that

$$\begin{aligned} \int_0^{\pi} \frac{\sin x \sin^3 nx}{\sin^6(1-n)x} dx &= n \int_0^{\pi} \frac{\sin nx \sin^3 x}{\sin^3(1-n)x} dx \\ &= \frac{n}{1-n} \int_0^{\pi} \frac{\sin x \sin nx \sin(1+n)x}{\sin^2(1-n)x} dx \dots\dots\dots (4), \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} \frac{\sin x \sin^3 nx}{\sin^3(1-n)x} dx &= n \int_0^{\pi} \frac{\sin nx \sin^3 x}{\sin^3(1-n)x} dx \\ &= \frac{n}{1-n} \int_0^{\pi} \frac{\sin x \sin nx \sin(1+n)x}{\sin^2(1-n)x} dx \dots\dots\dots (5), \end{aligned}$$

if in (3) $n > 0 < 2$, and in (4) $n > \frac{1}{2}$ and $< \infty$.]

6852. (By Professor CAVALLIN, M.A.)—Within a sphere a point P is taken at random, and two other points Q, R are taken at random, Q in the sphere concentric with the given sphere and passing through P, and R in the sphere concentric with the given sphere and touching PQ; prove that, if a denotes the radius of the given sphere, p the perpendicular from its centre on the plane determined by the points P, Q, R, and n any positive number, the average value of p^n is

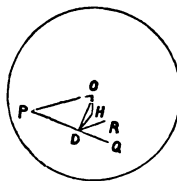
$$\frac{3 \cdot 5 \cdot 7 \cdot 9}{(n+1)(n+3)(n+5)(n+7)(n+9)} a^n.$$

Solution by Professor SEITZ, M.A.; G. HEPPLE, M.A.; and others.

Let OH be the perpendicular from O, the centre of the given sphere, on the plane PQR, and HD perpendicular to PQ. Then OD is perpendicular to PQ, and R is limited to the sphere whose radius is OD. Let $OP = x$, $PQ = y$, $DR = z$, $OH = p$, $\angle OPQ = \theta$, $\angle ODH = \phi$, and $\angle HDR = \psi$.

Then $OD = x \sin \theta$, $HD = x \sin \theta \cos \phi$, $p = x \sin \theta \sin \phi$; an element of the sphere at P is $4\pi x^2 dx$, at Q it is $2\pi y^2 \sin \theta d\theta dy$, and at R it is $x^2 \cos \psi d\psi dz$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ , 0 and $\frac{1}{2}\pi$; of x , 0 and a ; of y , 0 and $2x \cos \theta = y'$; of ψ , $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; and of z , 0 and $2x \sin \theta \cos \phi \cos \psi = z'$.



Hence, omitting constant factors common to numerator and denominator, we have, for the average value of p^n ,

$$\begin{aligned} & \frac{\int_0^{1\pi} \int_0^{1\pi} \int_0^a \int_0^{y'} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} p^n \sin \theta \, d\theta \, d\phi \, x^2 \, dy \, \cos \psi \, d\psi \, z^2 \, dz}{\int_0^{1\pi} \int_0^{1\pi} \int_0^a \int_0^{y'} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \sin \theta \, d\theta \, d\phi \, x^2 \, dy \, \cos \psi \, d\psi \, z^2 \, dz} \\ &= \frac{\int_0^{1\pi} \int_0^{1\pi} \int_0^a \sin^{n+4} \theta \cos^3 \theta \, d\theta \sin^n \phi \cos^3 \phi \, x^{n+8} \, dx}{\int_0^{1\pi} \int_0^{1\pi} \int_0^a \sin^4 \theta \cos^3 \theta \, d\theta \cos^3 \phi \, x^8 \, dx} \\ &= \frac{a^{n+9}}{n+9} \left(\frac{1}{n+1} - \frac{1}{n+3} \right) \left(\frac{1}{n+5} - \frac{1}{n+7} \right) = \frac{3 \cdot 5 \cdot 7 \cdot 9}{(n+1)(n+3)(n+5)(n+7)(n+9)} a^n. \\ &= \frac{a^9}{9} (1 - \frac{1}{3})(\frac{1}{3} - \frac{1}{5}) \end{aligned}$$

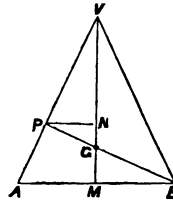
6814. (By Professor HUDSON, M.A.) — Prove (1) the following construction for the centre of gravity of a right circular cone:—let AB be a diameter of the base, V the vertex, divide VA, VB in P, Q, in the ratio 3 : 2, where BP, AQ intersect in the centre of gravity; and show (2) that the same construction holds for a cone on an elliptic base.

Solution by T. WOODCOCK, B.A.; E. RUTTER; and others.

Let PB meet the axis VM in G, and draw PN perpendicular to the axis; then $NM = \frac{1}{3} VM$; but $NG : GM = PN : BM = 3 : 5$,

$$\therefore GM : NM = 5 : 8, \quad \therefore GM = \frac{1}{3} VM.$$

Therefore (1) G is the centre of gravity of a right circular cone; and, since the proof merely requires that M should be the middle point of each diameter AB of the base, it is (2) applicable also to a right cone on an elliptic base.



6425. (By W. J. C. SHARP, M.A.)—Prove that one and only one quadric surface can be drawn through a given twisted quintic curve.

Solution by the PROPOSER.

This is somewhat too generally stated; it should stand, "not more than one quadric can be drawn." This is a case of the general statement that "in general at most one n -ic surface can be drawn through a given twisted $\frac{1}{2}n(n+3)$ -ic curve. For the curve cuts any arbitrary plane in $\frac{1}{2}n(n+3)$ points, which will in general determine one plane n -ic curve, and which must be the section of the n -ic surface if there be one. The exceptions are those noticed in SALMON'S *Higher Curves*, and will occur when the curve is the partial intersection of two n -ic surfaces.

7073. (By Professor SYLVESTER, F.R.S.)—If from each point of inflexion tangents are drawn to a cubic curve, meeting it in 27 points, and from each of these tangents meeting it in 108 points, and again from each of these tangents meeting it in 432 points, and so on; prove that the $27 \cdot 4^{i-1}$ points so obtained by applying this process i times will be the complete system of points in which proper curves of the order 2^i can be drawn having $3 \cdot 2^i$ consecutive points in common with the cubic.

Solution by W. J. C. SHARP, M.A.; BELLE EASTON; and others.

If a, b, c, d , &c. be the $3 \cdot 2^i$ successive points in which a 2^i -ic curve meets the cubic, these form an entire intersection; and therefore (SALMON'S *Higher Plane Curves*, Art. 157) so do the $3 \cdot 2^{i-1}$ successive points a', b', c', d' , &c., where ab, cd , &c. meet the cubic again, and these are the tangentials of the others. If this process be repeated i times, we arrive at three successive points forming an entire intersection, *i.e.*, at an inflexion.

7077. (By Professor MINCHIN, M.A.)—Prove that (1) the total energy of a rigid body is equal to the energy due to the velocity of translation of P + the energy due to the rotation round P , if P is any point on the quadric whose equation referred to the principal axes at G (the centre of mass) is $\lambda^2 + \mu^2 + \nu^2 + a\lambda + b\mu + c\nu = 0$, where (a, b, c) are the velocity components of G , and $\lambda \equiv \omega_x x - \omega_y y$ &c.; and (2) in uniplanar motion the locus of P is the circle described on GI as diameter ($I \equiv$ instantaneous centre). [A property always enunciated as if belonging to G exclusively.]

Solution by Prof. TOWNSEND, F.R.S.; T. WOODCOCK, B.A.; and others.

That, for a rigid body in motion, the entire kinetic energy should be equal to the sum of its components due to the separate motions of translation and rotation at any point P of the mass, we must have, if uvw and pqr be the components of the two motions at the point, xyz the relative coordinates of the point with respect to the centre of inertia of the body, and $\xi + \zeta, y + \eta, z + \zeta$ those of any element dm of its mass,

$$\begin{aligned} & \Sigma \{ [u + q(z + \zeta) - r(y + \eta)]^2 + \&c. \} dm \\ &= m [(u + qz - ry)^2 + \&c.] + \Sigma [(q\zeta - r\eta)^2 + \&c.] dm \\ &= m [(u + qz - ry)^2 + \&c. + (qz - ry)^2 + \&c.]; \end{aligned}$$

therefore, at once, for every point xyz of the mass fulfilling the condition, we must have

$$(u + qz - ry)(qz - ry) + \&c. = 0,$$

and therefore, &c., as regards the first part of the question.

The equation $(qz - ry)^2 + \&c. + u(qz - ry) + \&c. = 0$, expressing that the square of the distance of xyz from the line $x : y : z = p : q : r$ varies as its distance from the plane $u(qz - ry) + \&c. = 0$, shows consequently that the quadric in question is a circular cylinder passing through the line and touching the plane. Again, as the equation is also manifestly satisfied for every point xyz of the line

$$(u + qz - ry) : (v + rx - pz) : (w + py - qx) = p : q : r,$$

that is, of the axis of the instantaneous screw motion of the body, it shows consequently also that the cylinder passes through that line, and finally, as the plane of connection of the two lines

$x : y : z = p : q : r$ and $(u + qx - ry) : (v + rx - pz) : (w + pq - qx) = p : q : r$ is manifestly perpendicular to the plane

$$u(qz - ry) + v(rx - pz) + w(py - qx) = 0,$$

it appears from it, lastly, that the plane of the two lines in question is a diametral plane of the cylinder, and that the lines themselves are consequently diametrically opposite generators of the surface. From these general results for motion in three dimensions, the second part of the question follows of course at once for the particular case referred to in its statement.

7079. (By the EDITOR.)—Prove that, if

$$ax + cy : by + dz = ay + cz : bx + dx = az + cx : bx + dy,$$

then (1) each ratio $= a + c : b + d$, and (2) $\Sigma x^2 = \Sigma (yz)$.

Solution by Prof. WOLSTENHOLME, M.A.; KATH GALE; and others.

$$\text{If } \frac{ax + cy}{by + dz} = \frac{ay + cz}{bx + dx} = \frac{az + cx}{bx + dy},$$

$$\text{then each ratio} = \frac{(ax + cy) + (ay + cz) + (az + cx)}{(by + dz) + (bx + dx) + (bx + dy)} = \frac{(a + c)(x + y + z)}{(b + d)(x + y + z)},$$

$$\text{also} = \frac{(ax + cy) + \omega(ay + cz) + \omega^2(az + cx)}{(by + dz) + \omega(bx + dx) + \omega^2(bx + dy)} = \frac{(a + c\omega)(x + \omega y + \omega^2 z)}{(b\omega^2 + d\omega)(x + \omega y + \omega^2 z)},$$

if ω be one of the impossible cube roots of 1; and, similarly, each ratio

$$= \frac{(a + c\omega)(x + \omega^2 y + \omega z)}{(b\omega + d\omega^2)(x + \omega^2 y + \omega z)}.$$

Hence, either $x + y + z$, $x + \omega y + \omega^2 z$, or $x + \omega^2 y + \omega z = 0$, or $x^3 + y^3 + z^3 = 3xyz$;

and each of the three ratios $= \frac{a + c}{b + d}$, or $\frac{a\omega + c}{b + d\omega^2}$, or $\frac{a\omega^2 + c}{b + d\omega}$.

If all the symbols represent real quantities, we must have $x = y = z$,

and each ratio $= \frac{a + c}{b + d}$; or $x + y + z = 0$, and each ratio

$$= \frac{c}{b - d} = \frac{c - a}{b} = -\frac{a}{d}.$$

[Prof. WOLSTENHOLME remarks that, in the Mathematical Tripos Examination for 1857, Dr. CAMPION set the question to prove that $a^3 + b^3 + c^3 = 3abc$, if

$$\frac{a \sin^2 \theta + b \sin^2 \phi}{b \cos^2 \theta + c \cos^2 \phi} = \frac{b \sin^2 \theta + c \sin^2 \phi}{c \cos^2 \theta + a \cos^2 \phi} = \frac{c \sin^2 \theta + a \sin^2 \phi}{a \cos^2 \theta + b \cos^2 \phi}.$$

If we put each of the three ratios in Quest. 7079 = λ , we get at once

$$\begin{vmatrix} a, & c-\lambda b, & -\lambda d \\ -\lambda d, & a, & c-\lambda b \\ c-\lambda b, & -\lambda d, & a \end{vmatrix} = 0; \quad \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = 0,$$

or $a^3 + (c-\lambda b)^3 - \lambda^3 d^3 + 3\lambda a d (c-\lambda b) = 0$, which gives the three factors

$$a + c - \lambda b - \lambda d, \quad a + \omega (c - \lambda b) - \lambda \omega^2 d, \quad a + \omega^2 (c - \lambda b) - \lambda \omega d,$$

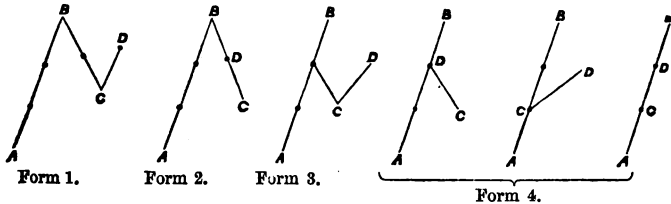
or gives for the three equal ratios the three values given above; the other determinant giving $x^3 + y^3 + z^3 = 3xyz$.]

6709. (By Dr. MACFARLANE, D.Sc., F.R.S.E.)—A is a great-grandchild of B, B is a grandparent of C, and C is a child of D. What is the relationship of A to D, expressed independently of B and C?

Solution by the PROPOSER.

The data are $A = c^3B$, $B = c^{-2}C$, $C = cD$. Hence $A = c^{3-2+1}D$; that is, A is a great-grandchild of a grandparent of a child of D. This is the general expression for the resulting relationship, and it can assume one or other of four irreducible forms. In the first place, the index $3-2$ may not cancel at all, or it may cancel once, giving $2-1$, or it may cancel twice, giving 1. Next, add on the $+1$ to each of these. In the first case, we have $3]-2+1$, which may not cancel, or it may cancel once; in the second case, we have $2]-1+1$, which may not cancel, or it may cancel once. In the third case, no cancelling is possible. Thus we obtain four, and only four, irreducible forms. Hence $A = c^{3-2+1}D$; or $A = c^3 \cdot^{-1}D$; or $A = c^{2-1+1}D$; or $A = c^2D$, where each relationship expression must be interpreted in its irreducible meaning. These equations mean that A is a grandchild of a brother-in-law or sister-in-law of D; or A is a grand-nephew or grandniece of D; or A is a grandchild of a consort of D; or A is a grandchild of D.

These forms are represented graphically in the accompanying figure, with the three varieties of the fourth form:—



A stroke of unit length in an upward direction represents c , and a similar stroke in a downward direction represents c^{-1} . The first form represents the general expression for the relationship, provided it be understood that two or more of the lines may coincide. In the case of a figure representing an irreducible relationship no pair of lines can coincide except the two lines be drawn as coincident.

7032. (By Professor JOHNSON, M.A.)—If three triangles have, when taken in pairs, a common axis of homology, prove that (1) the three centres of homology are in a straight line; (2) reciprocally, if three triangles have, when taken in pairs, a common centre of homology, the three axes of homology pass through a common point.

Solution by Prof. GENESE, M.A. ; Prof. MATZ, M.A. ; and others.

Project the axis of homology to infinity. The three triangles will now have their sides parallel. Let A_1, A_2, A_3 be three corresponding vertices; $m_1 : m_2 : m_3$ the ratios of the linear dimensions of the triangles. Then C_1 , the centre of homology of second and third triangles, lies on A_2A_3 , so that $A_2C_1 : A_3C_1 = m_2 : m_3$, and similarly for C_2, C_3 . Hence, by the theory of transversals, C_1, C_2, C_3 are in a straight line, &c.

6869. (By A. McINTOSH, M.A.)—Prove that (1) only two real tangents can be drawn from any point on the curve itself to a lemniscate; (2) the same line passes through the two real and the two imaginary points of contact of the four tangents that are analytically possible; (3) the following construction gives this line,—draw a radius-vector from the real node to the point from which the tangents are required to be drawn, produce it backwards one-half of its length, and through this extremity draw a line perpendicular to it, and this last will be the line sought; (4) hence a chord joining the points of contact of tangents drawn from any point on the curve touches an equilateral hyperbola, which has the same axis and the same foci as the lemniscate, which passes through the points of contact of the double tangents (which, of course, touches all the tangents that can be drawn from the three nodes to the curve); (5) to a lemniscate is drawn a tangent, which meets the curve again in two points, if a circle be drawn through the node and these two points, its centre lies on the lemniscate and its radius is equal to the radius vector from the node to the point of contact of the tangent; (6) if four tangents be drawn from any point (x', y', z') on the curve $ax^{-2} + by^{-2} + cz^{-2} = 0$ to touch the curve elsewhere, the four points of contact lie on the line $xx'^{-1} + yy'^{-1} + zz'^{-1} = 0$, which touches the conic $a^{-1}x^2 + b^{-1}y^2 + c^{-1}z^2 = 0$, which passes through the eight points of contact of bitangents to the curve; (7) the above trinodal quartic includes the case of the lemniscate.

Solution by the PROPOSER.

The equation of the lemniscate being $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, the equation of a tangent at any point (x', y') is

$$(x - x') [4x' (x'^2 + y'^2) - 2a^2x'] + (y - y') [4y' (x'^2 + y'^2) + 2a^2y'] = 0.$$

This may be reduced by the equation of the curve to

$$x [2x' (x'^2 + y'^2) - a^2x'] + y [2y' (x'^2 + y'^2) + a^2y'] = a^2(x'^2 - y'^2).$$

Hence the coordinates of the point of contact of a tangent from the point (x', y') on the curve to the curve are given by the equation

$$x' [2x (x^2 + y^2) - a^2x] + y' [2y (x^2 + y^2) + a^2y] = a^2(x^2 - y^2).$$

The coordinates of any point on the lemniscate may be represented by $a \frac{z+z^3}{1+z^4}$ and $a \frac{z-z^3}{1+z^4}$ where z is variable. Substitute, then, in the above equation for x and y , $a \frac{z+z^3}{1+z^4}$ and $a \frac{z-z^3}{1+z^4}$, and for x' and y' , $a \frac{z'+z'^3}{1+z'^4}$ and $a \frac{z'-z'^3}{1+z'^4}$ respectively, and after reducing the resulting equation and dividing by $(z-z')^2$, it will be found to be

$$z^4 z' + 2z^3 z'^2 + 2z + z' = 0 \text{ or } x(z'^{-1} + z') + y(z'^{-1} - z') + a = 0,$$

z being $a \frac{z+z^3}{1+z^4}$ and y , $a \frac{z-z^3}{1+z^4}$. This equation evidently denotes a line passing through the four points of contact of tangents from the points z' . It is perpendicular to the line $\frac{x}{z'+z'^3} = \frac{y}{z'-z'^3}$, which passes through the origin and the point z' ; also the distance of it from the origin is

$$\frac{a}{\left\{ 2 \left(\frac{1}{z'^2} + z'^2 \right) \right\}^{\frac{1}{2}}} = \frac{z'a 2^{\frac{1}{2}}}{2(1+z'^4)^{\frac{1}{2}}}.$$

But the distance of the point z' from the origin is

$$a \left\{ \left(\frac{z'+z'^3}{1+z'^4} \right)^2 + \left(\frac{z'-z'^3}{1+z'^4} \right)^2 \right\}^{\frac{1}{2}} = \frac{az' 2^{\frac{1}{2}}}{(1+z'^4)^{\frac{1}{2}}};$$

therefore the line joining the origin to the point from which the tangents are drawn is perpendicular to the line joining the points of contact, and if produced back to meet the latter line, the origin or node is a point of trisection. It is not difficult to see, now, that the point of intersection of these two lines lies on the lemniscate $(x^2+y^2)^2 = \frac{a^2}{4}(x^2-y^2)$, and the envelope of the line joining the points of contact, being the negative pedal of this lemniscate, is the hyperbola $x^2-y^2 = \frac{a^2}{4}$.

This equilateral hyperbola has the nodal or inflexional tangents for asymptotes, and its foci are at the distances $\frac{a}{\sqrt{2}}$ and $-\frac{a}{\sqrt{2}}$ from the origin, and therefore coincide with the foci of the lemniscate. The points in which it cuts the lemniscate are given by $r^2 = a^2 \cos 2\theta$ and $r^2 \cos 2\theta = \frac{a^2}{4}$, therefore $\frac{1}{\cos 2\theta} = 4 \cos 2\theta$, i.e., $\cos^2 2\theta = \frac{1}{4}$.

Hence $\cos 2\theta = \pm \frac{1}{2}$, therefore $2\theta = 60^\circ$ and $\theta = 30^\circ$, but this is the point of contact of a bitangent.

Consider the point of contact T of any tangent and the two points R and S in which it cuts the curve. Let N be the node. By what has just been proved, the line bisecting NR perpendicularly is parallel and similarly situated with respect to one of the loops to the line joining the points of contact of tangents from R to the curve. Hence it cuts the curve in points equally distant from the node, and consequently cuts it in one point the distance of which from the node is equal to NT. The same thing may be said of the line bisecting NS perpendicularly and therefore this line meets the curve where the line bisecting NR perpendicularly meets it. The

distance of this point, which is the centre of the circle circumscribing the triangle NRS, from the node is equal to NT.

To prove that only two real tangents can be drawn from a point on the curve, it is only necessary to remark that the invariant I of the biquadratic $z^2z' + 2z^2z'' + 2z + z' = 0$ vanishes. The reducing cubic of this equation, by EULER's method, may then be brought to the form $\lambda^3 + 2J = 0$. This has only one real root, therefore two roots of the biquadratic are real and two are imaginary. This is a point sufficiently evident from the figure of the curve, but it is interesting to see how it comes out analytically.

The equation of a tangent at the point (x', y', z') to the curve $\frac{a}{x^2} + \frac{b}{y^2} + \frac{c}{z^2} = 0$ may be put in the form $\frac{ax}{x'^2} + \frac{by}{y'^2} + \frac{cz}{z'^2} = 0$. Hence, if this line touches the curve at (x, y, z) and intersects it at (x', y', z') , the equation connecting the points will be $\frac{ax'}{x'^2} + \frac{by'}{y'^2} + \frac{cz'}{z'^2} = 0$.

Any point on the curve will be given (i being $= \sqrt{-1}$) by

$$\frac{aiz}{c^{\frac{1}{2}}x} = i \cos \theta \quad \text{and} \quad \frac{b^{\frac{1}{2}}z}{c^{\frac{1}{2}}y} = i \sin \theta.$$

Substituting then these values in the foregoing equation, together with

$$\frac{aiz'}{c^{\frac{1}{2}}x'} = i \cos \theta' \quad \text{and} \quad \frac{b^{\frac{1}{2}}z'}{c^{\frac{1}{2}}y'} = i \sin \theta', \quad \text{we easily get} \quad \frac{\cos^3 \theta}{\cos \theta'} + \frac{\sin^3 \theta}{\sin \theta'} = 1;$$

$$\therefore \cos^3 \theta \sin \theta' + \sin^3 \theta \cos \theta' = \cos \theta' \sin \theta' = \cos \theta' \sin \theta' (\cos^2 \theta + \sin^2 \theta).$$

$$\text{Hence} \quad \sin \theta' \cos^2 \theta (\cos \theta - \cos \theta') + \cos \theta' \sin^2 \theta (\sin \theta - \sin \theta') = 0.$$

$$\text{Hence} \quad \tan \theta' \cot^2 \theta \tan \frac{1}{2}(\theta + \theta') = 1,$$

$$\text{therefore} \quad \frac{\tan \frac{1}{2}(\theta + \theta')}{\tan \theta'} = \frac{\tan^2 \theta}{\tan^2 \theta'},$$

$$\frac{\sin \frac{1}{2}(\theta - \theta')}{\cos \frac{1}{2}(\theta + \theta') \cos \theta' \tan \theta'} = \frac{\sin(\theta - \theta') \sin(\theta + \theta')}{\cos^2 \theta \cos^2 \theta' \tan^2 \theta'},$$

$$1 = \frac{2 \cos \frac{1}{2}(\theta - \theta') \cos \frac{1}{2}(\theta + \theta') \sin(\theta + \theta')}{\cos^2 \theta \sin \theta'},$$

$$\cos^2 \theta \sin \theta' = \cos \theta \sin \theta \cos \theta' + \sin \theta' \cos^2 \theta + \sin \theta \cos^2 \theta' + \sin \theta' \cos \theta' \cos \theta.$$

$$\text{or} \quad \cos \theta \sin \theta + \sin \theta \cos \theta' + \sin \theta' \cos \theta = 0.$$

When this is transformed, by resubstituting for $\cos \theta$, $\sin \theta$, &c., in terms of x , y , z , &c., it becomes $\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0$.

This is the equation of a line which clearly passes through the four points of contact of four tangents from (x', y', z') to the curve.

Since also $\frac{a}{x'^2} + \frac{b}{y'^2} + \frac{c}{z'^2} = 0$, it touches the conic $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$, which passes through the eight points of contact of bitangents to the curve

$$\frac{a}{x^2} + \frac{b}{y^2} + \frac{c}{z^2} = 0.$$

This may also be proved without the use of trigonometrical functions.

Let $\frac{z}{x} = \lambda$ and $\frac{z}{y} = \mu$, with similar substitutions for $\frac{z'}{x'}$ and $\frac{z'}{y'}$.

Then we have for the tangent

$$\frac{a\lambda^2}{\lambda'} + \frac{b\mu^2}{\mu'} + c = 0 \dots\dots\dots(1).$$

Also $a\lambda^2 + b\mu^2 + c = 0$ and $a\lambda'^2 + b\mu'^2 + c = 0 \dots\dots\dots(2, 3).$

From (1), (2), and (2), (3),

$$a\lambda^2 \left(\frac{\lambda}{\lambda'} - 1 \right) = -b\mu^2 \left(\frac{\mu}{\mu'} - 1 \right), \quad a(\lambda^2 - \lambda'^2) = -b(\mu^2 - \mu'^2).$$

Dividing, we have $\frac{(\lambda + \lambda')\lambda'}{\lambda^2} = \frac{(\mu + \mu')\mu'}{\mu^2}.$

When this equation is cleared from fractions, it will be seen to be divisible by $\lambda'\mu - \lambda\mu'$, and the result will be $\lambda\mu + \lambda'\mu + \lambda\mu' = 0$, which comes to the same thing as before.

5149. (By S. TEBAT, B.A.)—A solid, having a plane base, rests on a smooth horizontal plane, and a heavy particle descends in a vertical plane passing through the centre of gravity in a curvilinear groove between two given points in the solid; find the brachystochrone.

Solution by the PROPOSER.

Let the higher point be the origin, m the mass of the particle, m' the mass of the solid, xy the coordinates of m at time t , x' the abscissa of the centre of gravity, and h the initial value of x' . Then, by the principle of

vis viva, $m \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} + m' \left(\frac{dx'}{dt} \right)^2 = 2gmy.$

But $mx + m'x' = m'h$; therefore, putting $1 + \frac{m'}{m} = n^2$, we have

$$n^2 \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = 2gy; \quad \text{therefore} \quad t = \int \left\{ \frac{p^2 + n^2}{2gy} \right\}^{\frac{1}{2}} dx,$$

and $V = \left\{ \frac{p^2 + n^2}{y} \right\}^{\frac{1}{2}} = Pp + \frac{1}{c}.$

But $P = \frac{p}{[y(p^2 + n^2)]^{\frac{1}{2}}}$; therefore $p^2 = \left(\frac{dy}{dx} \right)^2 = n^2 \left(\frac{c^2 n^2}{y} - 1 \right),$

and $nx = \frac{1}{2} c^2 n^2 \text{ vers}^{-1} \frac{2y}{c^2 n^2} - (c^2 n^2 y - y^2)^{\frac{1}{2}}.$

Since the solid has slipped back through the space $h - x' = \frac{m}{m'} x$, if we write $\frac{m}{m'} x + x = n^2 \frac{m}{m'} x$ for x , we get the equation to the groove, namely,

$$n^3 \frac{m}{m'} x = \frac{1}{2} c^2 n^2 \text{ vers}^{-1} \frac{2y}{c^2 n} - (c^2 n^2 y - y^2)^{\frac{1}{2}}.$$

If a, b be the coordinates of the second point, we have

$$n^3 \frac{m}{m'} a = \frac{1}{2} c^2 n^2 \text{ vers}^{-1} \frac{2b}{c^2 n^2} - (c^2 n^2 b - b^2)^{\frac{1}{2}},$$

which determines c .

7012. (By J. J. WALKER, M.A.)—Show that (1) the squares of tangents drawn from an external point (x', y') to an ellipse are, if $s' = b^2x'^2 + a^2y'^2 - a^2b^2$, given by the equation

$$(b^2x'^2 + a^2y'^2)^2 t^4 - 2[s'(x'^2 + y'^2) + b^4x'^2 + a^4y'^2] s' t^2 + [c^4 - 2c^2(x'^2 - y'^2) + (x'^2 + y'^2)^2] s'^2 = 0;$$

(2) the lengths t, t' of the tangents are given linearly by

$$(b^2x'^2 + a^2y'^2) t^2 - \frac{2a'b'}{k} s' t + (a'^2 + b'^2 - x'^2 - y'^2) s' = 0;$$

$$\text{viz., } t + t' = \frac{2a'b'}{k} \sin^2 \frac{1}{2}(\theta - \theta'), \quad t - t' = \frac{c^2 k}{a'b'} \sin(\theta + \theta') \tan \frac{1}{2}(\theta - \theta'),$$

where a', b' are the semi-axes of the confocal ellipse through (x', y') ; $k^2 = a'^2 - a^2 = b'^2 - b^2$; and θ, θ' are the excentric angles of the points of contact of the tangents t, t' respectively.

Solution by Rev. T. R. TERRY, M.A.; Prof. WOLSTENHOLME; and others.

1. From the usual quadratic for the length of the intercept

$$\rho^2 (a^2 \sin^2 \phi + b^2 \cos^2 \phi) + 2\rho (a^2 y' \sin \phi + b^2 x' \cos \phi) + s' = 0,$$

we see that for a tangent we must have

$$s' (a^2 \sin^2 \phi + b^2 \cos^2 \phi) = (a^2 y' \sin \phi + b^2 x' \cos \phi)^2 \dots\dots\dots(1),$$

$$a^2 \sin^2 \phi + b^2 \cos^2 \phi = \frac{s'}{t^2} \dots\dots\dots(2);$$

$$\therefore a^2 b^2 t^2 (a^2 y'^2 - b^2 x'^2) + c^2 s'^2 - a^4 y'^2 s' + b^4 x'^2 s' = 2a^2 b^2 x' y' c^2 t^2 \sin \phi \cos \phi,$$

whence squaring we get the first result.

2. If we denote this and the second result by

$$A^2 t^4 - 2B t^2 + C = 0, \quad \text{and} \quad A t^2 - 2B' t + C' = 0,$$

the conditions that the roots of the second equation should be the two positive roots of the first equation are $B = 2B'^2 - AC'$ and $C = C'^2$, which are readily proved to be true by means of the equation

$$k^4 + k^2 (a^2 + b^2 - x'^2 - y'^2) - s' = 0.$$

Again, since $s' = (b^2 x'^2 + a^2 y'^2) \sin^2 \frac{1}{2}(\theta - \theta')$, it follows from the second equation that $t + t' = \frac{2a'b'}{k} \sin^2 \frac{1}{2}(\theta - \theta')$; but directly from a figure we see that $t^2 - t'^2 = 2c^2 \sin^2 \frac{1}{2}(\theta - \theta') \sin(\theta + \theta') \tan \frac{1}{2}(\theta - \theta')$, whence the last result.

7054. (By Professor MALET, M.A.)—If from any point on the outer of two similar and coaxial ellipsoids a tangent cone be drawn to the inner, prove that the volume contained between the surfaces of the cone and either of the ellipsoids is constant.

Solution by T. WOODCOCK, B.A.; J. O'REGAN; and others.

The theorem is evidently true for two concentric spheres, and it is of the "projective" kind. Now two concentric spheres project orthogonally into two similar coaxial ellipsoids. Therefore, &c.

7068. (By W. J. C. SHARP, M.A.)—Show that, if n be any integer, $n^5 - n$ is divisible by 30, and by 240 when n is odd.

Solution by EMMA ESSENNELL; J. S. JENKINS; and others.

Since $n^5 - n = n(n-1)(n+1)(n^2+1)$, and n must be of the form $5m$, $5m \pm 1$, $5m \pm 2$; if we put either of these values for n in the given expression, either of the factors becomes a multiple of 5; next $(n-1)n(n+1)$ three consecutive integers must be divisible by 6; thus $n^5 - n$ is divisible by $6 \times 5 = 30$.

Next, it will be sufficient to show that, if n is odd, $n^5 - n$ is divisible by 16, having already shown that in every case it is divisible by 15.

Put $n = 2m + 1$; we obtain $8(2m+1)m(m+1)(2m^2+2m+1)$; $m(m+1)$ is evidently divisible by 2, therefore $n^5 - n$ is divisible by 16 when n is odd, and thus by $16 \times 15 = 240$.

4784. (By H. HART.)—If ABC be a spherical triangle, and AT the tangent arc to the circumscribing small circle, prove that the angle BAT is equal to the angle C in the alternate segment minus half the spherical excess of the triangle.

Solution by J. HAMMOND, M.A.; BELLE EASTON; and others.

Let E_1, E_2, E_3 denote respectively the spherical excess of the triangles BOC, COA, AOB; then

$$E_1 + E_2 + E_3 = E,$$

the spherical excess of ABC; $BAT + OAB = \frac{1}{2}\pi$;

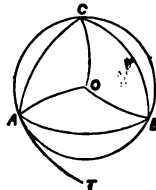
$$2OAB + AOB = \pi + E_3, \quad 2OCA + AOC = \pi + E_2,$$

$$2OCB + BOC = \pi + E_1.$$

Whence $2OAB + 2C + 2\pi = 3\pi + E$;

therefore $OAB = \frac{1}{2}\pi - C + \frac{1}{2}E$,

and $BAT = C - \frac{1}{2}E$.



6841. (By Rev. A. J. C. ALLEN, M.A.)—From a point O a straight line OA is drawn, making an angle α ($< \frac{\pi}{n+1}$) with a fixed straight line AB, and n other straight lines $OA_1, OA_2, \dots, OA_n$ are drawn to it, making the angles $\angle AOA_1, \angle AOA_2, \dots$ all equal and each $= \alpha$; if $R_1, R_2, \dots, R_n$ be the radii of the circles circumscribing the triangle OAA_1, OA_1A_2 ; prove that, if a be the perpendicular from O on AA_n , then

$$R_1 + R_2 + \dots + R_n = - \frac{a \sin n\alpha}{\sin 2\alpha \cos (n+1)\alpha}.$$

Solution by T. WOODCOCK, B.A. ; G. M. REEVES, M.A. ; and others.

We have

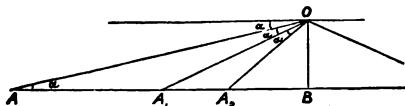
$$R_1 = \frac{1}{2} AA_1 \operatorname{cosec} \alpha,$$

$$R_2 = \frac{1}{2} A_1 A_2 \operatorname{cosec} \alpha,$$

and so on,

$$\therefore R_1 + R_2 + \dots + R_n = \frac{1}{2} AA_n \operatorname{cosec} \alpha$$

$$= \frac{1}{2} a \{ \cot \alpha - \cot (n+1) \alpha \} \operatorname{cosec} \alpha = \frac{-a \sin n\alpha}{\sin 2\alpha \cos (n+1) \alpha}.$$



7063. (By R. F. SCOTT, M.A.)—A and B are partners in a business in which their interests are in the ratio of a to b . They admit C to the partnership (without altering the capital) in such a way that the interests of the three partners in the business are then equal. C pays £ c for the privilege; how is this to be divided between A and B?

Solution by EMMA ESSENNELL ; E. RUTTER ; and others.

The capital is evidently £ $3c$, and therefore the values of A's and B's shares at first are £ $\frac{3ac}{a+b}$ and £ $\frac{3bc}{a+b}$ respectively, and the sums they should receive £ $\left(\frac{3ac}{a+b} - c \right)$ and £ $\left(\frac{3bc}{a+b} - c \right)$, which are as $2a-b : 2b-a$. If, therefore, a exceed $2b$, B will have to pay a sum of £ $\frac{a-2b}{a+b} c$ instead of receiving anything.

6683. (By G. F. WALKER, M.A.)—If n particles of masses $m_1, m_2 \dots m_n$ respectively, are rigidly connected together, and are capable of motion one on each of a series of smooth concentric circles in a vertical plane of radii $a_1, a_2 \dots a_n$ respectively; show that the time of a small oscillation about the position of stable equilibrium under the action of gravity is

$$2\pi \left(\frac{\sum m_r a_r^2}{g (\sum m_r^2 a_r^2 + 2S m_r m_s a_r a_s \cos \theta_{rs})} \right)^{\frac{1}{2}},$$

where Σ denotes summation from $r = 1$ to $r = n$, and S denotes summation for all different pairs of values of r and s , and θ_{rs} is the angle subtended at the common centre by the line joining m_r and m_s .

Solution by CHRISTINE LADD, B.A. ; the PROPOSER ; and others.

Let θ be the angle the line joining m_1 to the centre makes with the vertical, and let α_r be the angle subtended at the centre by the line join-

ing m_r to m_{r+1} , and let s_r denote $a_1 + a_2 + \dots + a_{r-1}$. Equation of motion is

$$\Sigma m \left(a \frac{d\theta}{dt} \right)^2 = C + 2g \Sigma m a \cos(\theta + s), \therefore \Sigma m a^2 \frac{d^2\theta}{dt^2} = -g \Sigma m a \sin(\theta + s),$$

Position of equation is given by $\Sigma m a \sin(\theta_0 + s) = 0$, and consequently

$$\Sigma m a^2 \frac{d^2\phi}{dt^2} = -g \Sigma m a \cos(\theta_0 + s) \phi = -g \{ [\Sigma m a \cos s]^2 + [\Sigma m a \sin s]^2 \}^{\frac{1}{2}} \phi,$$

and required time $= 2\pi \left(\frac{\Sigma m a^2}{g \{ [\Sigma m a \cos s]^2 + [\Sigma m a \sin s]^2 \}^{\frac{1}{2}}} \right)^{\frac{1}{2}} = \&c.$, as given.

6786. (By ASUTOSH MUKHOPADHYAY.)—Prove (1) geometrically that the rectangle contained by the perimeters of an acute-angled triangle and its pedal triangle, is equal to twice the sum of the rectangles contained by the perpendiculars, two and two; and (2) investigate whether the proposition holds for obtuse-angled triangles.

Solution by D. EDWARDES, B.A.; BELLE EASTON; and others.

The lines joining the vertices of a triangle to the centre of circumscribed circle are perpendicular to the sides of the pedal; therefore $\Delta = \frac{1}{4}R$ (perimeter of pedal triangle). Again, $\Sigma p_1 p_2 = \frac{\Delta^2}{abc}$ (perimeter of original triangle), and $R = abc + 4\Delta$, therefore $\&c.$

6741. (By Professor HOOVER.)—Show that the average area of circles whose diameters are the focal chords of an ellipse of semi-axes a, b , is $\frac{b\pi}{2a}(a^2 + b^2)$.

Solution by KATE GALE; D. EDWARDES; and others.

$$\begin{aligned} \text{Required average} &= \int_0^{1\pi} \pi \rho^2 d\theta + \int_0^{1\pi} d\theta \text{ and } \rho = \frac{l}{1 - e^2 \cos^2 \theta}, \\ \text{therefore } \Delta &= 2l^2 \int_0^{1\pi} \frac{d\theta}{(1 - e^2 \cos^2 \theta)^2} = 2l^2 \int_0^\infty \frac{(1 + z^2) dz}{(1 - e^2 + z^2)^2} \\ &= 2l^2 \int_0^\infty \frac{dz}{1 - e^2 + z^2} + 2e^2 l^2 \int_0^\infty \frac{dz}{(1 - e^2 + z^2)^2} = \frac{2l^2}{(1 - e^2)^{\frac{1}{2}}} \left[\tan^{-1} \frac{z}{(1 - e^2)^{\frac{1}{2}}} \right]_0^\infty \\ &\quad + \frac{2e^2 l^2}{2(1 - e^2)^{\frac{1}{2}}} \left[\frac{1}{1 - e^2} \times \tan^{-1} \frac{z}{(1 - e^2)^{\frac{1}{2}}} + \frac{z}{(1 - e^2)^{\frac{1}{2}}(1 - e^2 + z^2)} \right]_0^\infty \\ &= \frac{b^2 \pi}{a} + \frac{e^2 a b \pi}{2} = \frac{b\pi}{2a}(a^2 + b^2). \end{aligned}$$

700? (By Prof. NASH, M.A.)—The reciprocal polar of a given central conic with respect to a circle is similar to the given conic; prove that (1) the locus of the centre of the circle is a bicircular quartic having a node at the centre, and foci coinciding with the foci of the conic; (2) if the given conic is an ellipse, the nodal tangents are the equi-conjugate diameters; (3) if the conic is an hyperbola, the node is a conjugate point; (4) in the latter case the eccentricity of the hyperbola must be $< 2\frac{1}{2}$ for the locus to be a real curve.

Solution by Prof. TOWNSEND, F.R.S.; Prof. MATZ, M.A.; and others.

The asymptotic angle of the reciprocal polar of a conic with respect to a circle being equal to that determined by the two tangents to the conic from the centre of the circle, the required locus, for a given central conic of semi-axes a and b , is consequently that of the intersections of its tangents at the angle of its asymptotes; the equation of which is found immediately as follows:—

Since, for tangents intersecting at any constant angle ϕ , the equation of the locus of their intersections (see SALMON'S *Conic Sections*, 5th edit., art. 169) is $[(x^2 + y^2) - (a^2 + b^2)]^2 \tan^2 \phi = 4[b^2x^2 + a^2y^2 - a^2b^2]$, which represents in all cases a bicircular quartic having quadruple contact with the conic at the four imaginary points of its intersection with its orthocycle, the four tangents at which to both curves intersect in pairs at the two circular points at infinity and at the four foci of the conic, which latter in consequence are also foci of the quartic; therefore, for tangents intersecting at the asymptotic angle of the conic, for which

$$(a^2 + b^2)^2 \tan^2 \phi + 4a^2b^2 = 0,$$

the equation is

$$a^2b^2[(x^2 + y^2) - (a^2 + b^2)]^2 + (a^2 + b^2)^2[b^2x^2 + a^2y^2 - a^2b^2] = 0,$$

or its obvious equivalent on reduction

$$a^2b^2(x^2 + y^2)^2 + (a^4 - b^4)(a^2y^2 - b^2x^2) = 0,$$

which shows that in that case the locus has, in addition, a double point at the origin, the tangents at which are given by the equation $a^2y^2 - b^2x^2 = 0$, and are consequently real for the ellipse and coincide with the equiconjugate diameters of the curve, and imaginary for the hyperbola; and is itself real or imaginary in the latter case, according as b^2 , which is then negative, is less or greater in absolute magnitude than a^2 , that is, according as $(a^4 - b^4)$ —see last equation—is positive or negative. And therefore, &c., as regards the several parts of the equation.

7108. (By the EDITOR.)—Into a full conical wine-glass whose depth is a and generating angle α , there is dropped a spherical ball that causes the greatest overflow; show that (1) the radius of the ball is $\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}$, and (2) when $\alpha = 30^\circ$, the radius of the ball is $\frac{1}{4}a$, and its centre is at the middle of the top of the wine-glass.

Solution by J. W. SHARPE, M.A. ; SARAH MARKS ; and others.

1. Let $OE = OF = OK =$ radius of the ball $= x$; $OD = y$; $DK = x + y =$ length of the vertical diameter below the mouth of the cone; $V =$ volume of the ball below this level; then we have

$$V = \frac{2}{3}\pi x^3 + \pi y(x^2 - \frac{1}{3}y^2) \text{ and } x \operatorname{cosec} \alpha + y = a,$$

If V be a maximum or a minimum, we have, by differentiation,

$$6x(x+y)dx + 3(x^2 - y^2)dy = 0$$

and $dx + \sin \alpha dy = 0$; hence for one condition $x + y = 0$, which corresponds to the case of the ball just touching the surface AB of the fluid,—a case excluded by the conditions of the question.

Rejecting this factor, we get $2xdx + (x - y)dy = 0$, and $dx + \sin \alpha dy = 0$.

Eliminating the ratio $dy : dx$ we obtain

$$2x \sin \alpha - x + y = 0, \text{ where } y = a - x \operatorname{cosec} \alpha; \text{ hence } x = \frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}.$$

2. When $\alpha = 30^\circ$, we have $x = \frac{1}{2}a$, and $y = 0$.



6195. (By Professor COCHEZ.)—Démontrer que le lieu des foyers des hyperboles ayant un sommet donnée et une asymptote également donnée est une Strophoïde droite.

Solution by Prof. SCOTT, M.A. ; W. J. C. SHARP, M.A. ; and others.

If the given asymptote be taken as axis of x , the perpendicular from the given vertex upon it as axis of y , so that the vertex is $(0, k)$; and if $y = \tan 2\alpha (x + c)$ be the other asymptote, so that $(-c, 0)$ is the centre, then $y = \tan \alpha (x + c)$ is the axis, and $k = c \tan \alpha$ and at the focus $x + c = c \sec \alpha$ and $y = \tan \alpha (x + c)$; therefore $\cos \alpha = \frac{k}{y}$ and $x \sin \alpha = k(1 - \cos \alpha)$; therefore $x^2(y + k) = k^2(y - k)$ or $(x^2 - k^2)(y + k) + 2k^3 = 0$, the locus required, a cubic with three inflexional asymptotes, and no node.

7018. (By W. R. WESTROFF ROBERTS, M.A.)—Determine the locus of the point of contact of a geodesic tangent on an ellipsoid, touching a fixed line of curvature, with a series of sphero-conics on the same surface.

Solution by Professor TOWNSEND, F.R.S.

Denoting by O the centre of the ellipsoid, by abc its axes of figure, by δ

the parameter of the fixed line of curvature, by xyz the coordinates of the point of contact R with the variable sphero-conic of any of the four geodesics touching it and the line, by r the radius OR, by p the perpendicular OP from O on the tangent plane at R, by q the distance PR, by s the radius OS perpendicular to the plane POR and therefore parallel to the sphero-conic tangent at R, by $\alpha\beta\gamma$ the direction angles of OP, and by $\lambda\mu\nu$ those of OS; then, since $q \cos \lambda = (y \cos \gamma - z \cos \beta) = p (c^{-2} - b^{-2}) y$, with similar values for $q \cos \mu$ and for $q \cos \nu$, and since

$$s^{-2} = a^{-2} \cos^2 \lambda + b^{-2} \cos^2 \mu + c^{-2} \cos^2 \nu,$$

therefore, at once,

$$s^2 [a^{-2} (c^{-2} - b^{-2})^2 y^2 z^2 + \&c.] = [(c^{-2} - b^{-2})^2 y^2 z^2 + \&c.];$$

hence, multiplying by p^2 , substituting for $p^2 s^2$ its constant value $a^2 b^2 c^2 + \delta^2$ for every geodesic touching the line of curvature of which δ is the parameter (SALMON'S *Geometry of Three Dimensions*, 3rd edit., art. 416), for p^{-2} its known equivalent $a^{-4} x^2 + b^{-4} y^2 + c^{-4} z^2$, and for $a^2 b^2 c^2$ its equivalent from the equation of the surface $b^2 c^2 x^2 + c^2 a^2 y^2 + a^2 b^2 z^2$, and clearing of fractions, we get, between the coordinates xyz of every point R of the required locus, the homogeneous equation

$$\begin{aligned} [b^4 c^4 x^2 + c^4 a^4 y^2 + a^4 b^4 z^2] [a^2 (b^2 - c^2)^2 y^2 z^2 + \&c.] \\ = \delta^2 [b^2 c^2 x^2 + c^2 a^2 y^2 + a^2 b^2 z^2] [a^4 (b^2 - c^2)^2 y^2 z^2 + \&c.], \end{aligned}$$

which is that of a sextic cone, passing doubly through the three axes of the ellipsoid, and touching doubly in its three principal planes the quadric cone connecting the line of curvature with the centre of the surface.

For, a complete line of curvature, consisting of course of a twin pair of opposite ovals, the complete locus in question, which with its determining cone is evidently symmetrical with respect to the three principal planes of the surface, is in general doubly tripartite in form; consisting, as the rule, of a twin pair of acnodal points at the extremities of the axis of the surface internal to the cone of connection of the ovals, of a twin pair of dumb-bell lemniscates circumscribed to the two ovals at their two tetrads of vertices, and of a twin pair of undulating reentrants intersecting quadruply at the intersections of the remaining two axes of the surface. The two real cyclic planes passing through the mean axes of the ellipsoid, separating the opposite twin regions of the reentrants, which always pass through the extremities of that axis, from those of the remaining two twin parts of the curve, which always lie in those of the twin ovals of the line.

Putting, successively, $x^2 = a^2$, $y^2 = b^2$, $z^2 = c^2$, in the preceding equation of the determining cone, and taking the quadratic parts only of the resulting equations in the remaining two variables; we get, for the three pairs of directions in which the curve passes doubly through the three pairs of opposite vertices of the ellipsoid, the three equations

$$\begin{aligned} c^4 (a^2 - b^2)^2 (b^2 - \delta^2) y^2 + b^4 (a^2 - c^2)^2 (c^2 - \delta^2) z^2 &= 0, \\ a^4 (b^2 - c^2)^2 (c^2 - \delta^2) z^2 + c^4 (b^2 - a^2)^2 (a^2 - \delta^2) x^2 &= 0, \\ b^4 (c^2 - a^2)^2 (a^2 - \delta^2) x^2 + a^4 (c^2 - b^2)^2 (b^2 - \delta^2) y^2 &= 0, \end{aligned}$$

which show that the three pairs of parallels to them through the centre are the three corresponding principal sections of the quadric cone whose

$$\text{equation is } \frac{(a^2 - \delta^2)}{a^4 (b^2 - c^2)^2} x^2 + \frac{(b^2 - \delta^2)}{b^4 (c^2 - a^2)^2} y^2 + \frac{(c^2 - \delta^2)}{c^4 (a^2 - b^2)^2} z^2 = 0,$$

and which breaks up into a pair of planes, real or imaginary, when $\delta^2 = a^2$

or b^2 or c^2 , that is, when the two ovals of the line of curvature come together and coincide with a principal section of the ellipsoid.

Hence, when $\delta^2 = b^2$, in which case alone the two planes of the pair are real, the two lemniscates of the locus circumscribed to the two ovals of the line become two figures of eight, intersecting at right angles the principal ellipse ac at the four real umbilici of the surface, and having their nodal centres at the extremities of its least axis. In the same case, the two reentrants of the locus, while passing through the extremities of the mean axis at the angle of intersection of the two planes, have common tangents parallel to that axis at their passages through the extremities of the greatest axis of the surface.

Making, successively, $b^2 = c^2$, $c^2 = a^2$, $a^2 = b^2$, in the general equation of the determining cone; it assumes, in the three cases respectively, the three simplified forms:—

$$\begin{aligned}(y^2 + z^2) x^2 [a^2 (a^2 - \delta^2) (y^2 + z^2) + b^2 (\delta^2 - \delta^2) x^2] &= 0, \\(x^2 + z^2) y^2 [\delta^2 (b^2 - \delta^2) (x^2 + z^2) + c^2 (c^2 - \delta^2) y^2] &= 0, \\(x^2 + y^2) z^2 [c^2 (c^2 - \delta^2) (x^2 + y^2) + a^2 (a^2 - \delta^2) z^2] &= 0,\end{aligned}$$

which show, for a surface of revolution, that the determining cone breaks up into a cylinder of evanescent radius round the polar axis, a pair of parallel planes coinciding with the equatorial plane, and the cone intersecting the surface in the line of curvature; and that the locus itself consists, in consequence, of the two poles of the surface, of the equatorial circle taken doubly, and of the two parallel circles constituting the two ovals of the line; which constitute respectively, for that case, the two anodal points, the two undulating reentrants, and the two dumb-bell lemniscates, of the general case.

6777. (By Dr. MACFARLANE, F.R.S.E.)—Find all the meanings which the term *first cousin* may have in accordance with the English laws of marriage.

Solution by the PROPOSER.

The general relationship expressed by *first cousin* is denoted by c^{2-2} .

$$\begin{aligned}\text{Now } \Sigma c^{2-2} A &= \Sigma c (m+f) c (m+f) c^{-1} (m+f) c^{-1} A, \\&= \Sigma \{ c m c m c^{-1} m c^{-1} + c m c m c^{-1} f c^{-1} + c m c f c^{-1} m c^{-1} + c m c f c^{-1} f c^{-1} \\&\quad + c f c m c^{-1} m c^{-1} + c f c m c^{-1} f c^{-1} + c f c f c^{-1} m c^{-1} + c f c f c^{-1} f c^{-1} \} A \\&= \Sigma (c^{2-2})^1 + 2 \Sigma (c^{2-2})^2 + 3 \Sigma (c^{2-2})^3 + \dots + 8 \Sigma (c^{2-2})^8;\end{aligned}$$

where 1 denotes that one and one only of the forms occurs in the person; 2, two and two only, etc.

For the sake of shortness, let the specific relationships be denoted by 1, 2, 3, 4, 5, 6, 7, 8, in order,

$$\begin{aligned}\Sigma (c^{2-2})^1 &= \Sigma \{ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \}, \\ \Sigma (c^{2-2})^2 &= \Sigma \{ 12 + 13 + 14 + 15 + 16 + 17 + 18 + 23 + 24 + 25 + 26 + 27 + 28 \\&\quad + 34 + 35 + 36 + 37 + 38 + 45 + 46 + 47 + 48 \\&\quad + 56 + 57 + 58 + 67 + 68 + 78 \}.\end{aligned}$$

An examination of the first seven of these terms will suffice to determine all the non-existent terms,

$$12 = c mc mc^{-1} mc^{-1} . c mc mc^{-1} fc^{-1} = c mc (mc^{-1} mc^{-1} . mc^{-1} fc^{-1}),$$

(by § 10 of Article). This reduces the question to whether the relationship within the bracket is necessarily non-existent by the laws of marriage. The question is, whether the father of the father of a person can be the father of the mother of the person. By Rule (§ 17 of Article), this is reduced to whether $m . mc^{-1} c fc mc^{-1} m$ is non-existent; that is, whether a man can be the husband of a daughter of his father.

13 reduces to $c \{ mc mc^{-1} . mc fc^{-1} \} mc^{-1}$, which depends on whether $mc mc^{-1} . mc fc^{-1}$ is non-existent. By the Rule referred to above, this is reduced to the question whether $m . mc^{-1} c fc^{-1} m cm$ is non-existent; that is, whether a man can be the husband of the mother of his son.

$14 = c \{ mc mc^{-1} mc^{-1} . mc fc^{-1} fc^{-1} \}$. The ultimate question is, whether $m . mc^{-1} c fc fc^{-1} mc mc^{-1} m$ can exist; that is, whether a man can be the husband of a daughter of the wife of his father.

$15 = (c mc . c fc) mc^{-1} mc^{-1}$. Reduces as in the case of 12.

$16 = c mc mc^{-1} mc^{-1} . c fc mc^{-1} fc^{-1}$. Reduces to

$$m . mc^{-1} c fc mc^{-1} fc^{-1} c mc mc^{-1} m,$$

that is, whether a man can be the husband of a daughter of the father of the wife of a son of his father.

$17 = (c mc mc^{-1} . c fc fc^{-1}) mc^{-1}$. Reduces to

$$m . mc^{-1} c fc fc^{-1} mc mc^{-1} m;$$

that is, whether a man can be the husband of a daughter of the wife of his father.

$18 = c mc mc^{-1} mc^{-1} . c fc fc^{-1} fc^{-1}$. Reduces to

$$m . mc^{-1} c fc^{-1} fc^{-1} c mc mc^{-1} m;$$

that is, whether a man can be the husband of the mother of the wife of a son of his father.

Of these questions only that of 12 and 15 is answered in the negative by the English law.

Hence the terms 12 and 15 are non-existent, and by the symmetry of the law, 34, 56, 78, and 26, 37, 48. The relationships of the third degree are rendered non-existent, only if they contain a non-existent relationship of the second degree. Hence

$$\Sigma (c^2-2)^3 = \Sigma \{ 136 + 138 + 146 + 147 + 167 + 168 + 235 + 238 + 245 + 247 \\ + 257 + 258 + 358 + 368 + 457 + 467 \}.$$

$$\text{And} \quad \Sigma (c^2-2)^4 = \Sigma \{ 1368 + 1467 + 2358 + 2457 \}.$$

All relationships of a higher degree are non-existent. Thus a person's first cousins fall into 48 classes; these sub-divided according to sex fall into 96 classes; and if the sex of the person who is the origin of the relationship be taken into account, the word is susceptible of 192 meanings.

6860. (By Professor MATZ, M.A.)—If the curve $\rho = m \sin 2\theta$ be inscribed in the ellipse $\rho^2 = b^2 + (1 - e^2 \cos^2 \theta)$; prove that

$$3e^2 \sin^2 \theta = (1 - e^2 + e^4)^{\frac{1}{2}} + 2e^2 - 1.$$

Solution by E. BUTTER; A. McMURCHY; and others.

Let O be the centre of the circumscribing circle, and P the orthocentre; then $OA = R$, $AP = 2R \cos A$, and $\angle OAP = C - B$; hence

$$\begin{aligned} OP^2 &= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B) \\ &= 5R^2 - 4R^2 \sin^2 A + 2R^2 - 4R^2 \sin^2 B + 2R^2 - 4R^2 \sin^2 C = \&c. \end{aligned}$$

7096. (By R. KNOWLES, B.A.)—If m, n are positive integers, prove that

$$\begin{aligned} &\frac{2^n}{m+1} - \frac{n \cdot 2^{n-1}}{(m+1)(m+2)} + \frac{n(n-1)2^{n-2}}{(m+1)(m+2)(m+3)} \dots \\ &\dots + (-1)^n \frac{n(n-1)(n-2) \dots 1}{(m+1)(m+2) \dots (m+n+1)} \\ &= \frac{1}{m+1} + \frac{n}{m+2} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{m+3} \dots + \frac{1}{m+n+1}. \end{aligned}$$

Solution by Rev. J. L. KITCHIN, M.A.; CHRISTINE LADD, B.A.; and others.

If the part under the integral sign of $\int_0^1 x^m (1+x)^n$ be expanded by the binomial theorem, we get the right-hand series; and if we integrate the same by parts, we get the left-hand series; hence the equality.

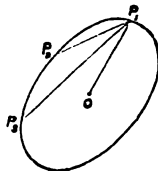
7022. (By Rev. H. G. DAX, M.A.)—In a closed convex curve, if m be the average area of a triangle one of whose angular points is the centre G, the two others being taken at random in the area; m_1 the average area when all three are taken at random; k_1, k_2 the principal semi-axes of gyration; and S the area; prove that $m_1 = \frac{2}{3}m + \frac{k_1^2 k_2^2}{2S}$.

Solution by the PROPOSER; BELLE EASTON; and others.

Let $S^2 u = \int_0^{2\pi} \int_0^{\theta} r^3 r_1^3 \sin(\theta - \theta_1) d\theta_1 d\theta$,
CG being the origin. Then, obviously, $m = \frac{2}{3}u$.

Also $v = \iint r_1^4 r^4 \sin^2(\theta - \theta_1) d\theta d\theta_1 = 32k_1^2 k_2^2 S^2$.

Now $48S^3 m_1 = \iiint K [r_2^3 d\theta_2 - r_1 d r_2 \sin(\theta_2 - \theta_1)]$
 $\times [r_2^3 d\theta_2 - r_1 d r_2 \sin(\theta_2 - \theta_1)] [r_1^3 d\theta_1],$



where $K = r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) - r_1 r_3 \sin(\theta_3 - \theta_1)$,
the integral being taken with the positive sign as long as θ_3 lies between
 θ_1 and θ_2 , and with the negative in other cases.

This, integrated by ordinary rules, gives

$$48S^2m_1 = 8S^2u + \frac{1}{2}v, \text{ or } m_1 = \frac{1}{2}m + \frac{k_1^2 k_2^2}{2S}.$$

6973. (By Rev. T. W. OPENSHAW, M.A.)—On a focal chord of a parabola as diameter is described a circle cutting the parabola again in P, Q; prove that the circle PSQ will touch the parabola.

Solution by E. W. SYMONS, M.A.; R. KNOWLES, B.A.; and others.

Let $y - mx + am = 0$ be a focal chord of $y^2 = 4ax$; then a circle on this will take the form $(y - mx + am)(y + mx + d) = \lambda(y^2 - 4ax)$(1).
The condition for a circle gives $\lambda = 1 + m^2$; and the ordinate of centre found from combining $y - mx + am = 0$ and $y^2 = 4ax$, is $\frac{2a}{m} = \frac{am + d}{2m^2}$ [found from (1)]; whence $d = 3am$. Again, a circle through P, Q takes the form $(y + mx + 3am)(y - mx + c) = (1 + m^2)(y^2 - 4ax)$; if this passes through $(a, 0)$, we find $c = -\frac{a}{m}$; hence the other chord of intersection is $y - mx - m^{-1}a = 0$, which is a tangent to $y^2 = 4ax$ at $(\frac{a}{m^2}, \frac{2a}{m})$, i.e., the vertex of the diameter which bisects the focal chord.

[This theorem forms Ex. 941 in Prof. WOLSTENHOLME'S *Math. Problems.*]

6690. (By J. HAMMOND, M.A.)—Prove that the limiting value, when $m = 0$, of

$$\frac{1}{\Gamma(m)} \int_0^1 e^{xy} \frac{dy}{(1-y)^{1-m}} = e^x.$$

Solution by D. EDWARDS; J. O'REGAN; and others.

Making the substitution $y = \frac{z}{1+z}$, we have

$$\begin{aligned} \int_0^1 e^{xy} \frac{dy}{(1-y)^{1-m}} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_0^{\infty} \frac{z^n dz}{(1+z)^{1+m+n}} = \sum_{n=0}^{\infty} \frac{x^n}{n!} B(n+1, m) \\ &\quad \text{(WILLIAMSON'S Integral Calculus, p. 157)} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\Gamma(n+1) \Gamma(m)}{\Gamma(n+m+1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \end{aligned}$$

dividing each term by $\Gamma(m)$ and putting $m = 0$.

7023. (By Rev. W. A. WHITWORTH, M.A.)—Four different rectangular parallelepipeds on square bases have their diagonals equal, and all their edges commensurable with the diagonal; find the solution in lowest integers.

Solution by G. HEPPLE, M.A.; CHARLOTTE A. SCOTT; and others.

Let a, b , be the edges of one such parallelepiped, and d the diagonal; then $d^2 = 2a^2 + b^2$. Let $b = d - e$; then $2de = 2a^2 + e^2$; hence e must be even. Let $e = 2f$; then $d = \frac{a^2}{2f} + f$; hence a must be even. Let $a = 2g$, then $d = \frac{2g^2}{f} + f$; hence, taking any value for g , and separating $2g^2$ into all possible pairs of factors f and h , we get

$$d = h + f, \quad a = 2g, \quad b = h - f.$$

For example, suppose $g = 6$, the factors of 72 are 72, 1; 36, 2; 24, 3; 18, 4; 12, 6; 9, 8; and we have $a = 12$; $b = 71, 34, 21, 14, 6, 1$; $d = 73, 38, 27, 22, 18, 17$.

By taking successive values of g , we find that the first instances of d being repeated four times are in the following sets:—

8	31	22	18	21	33	20	17	33	22	11	33
10	49	51	14	47	51	34	17	51	36	3	51
30	69	81	36	63	81	54	27	81	56	17	81

The next instance gives a set of six,

14	97	99	24	93	99	44	77	99	54	63	99	60	51	99	66	33	99.
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7094. (By D. EDWARDES.)—Prove that the area of a triangle in terms of the radii (ρ_1, ρ_2, ρ_3) of the escribed circles of its orthocentric triangle is

$$\frac{1}{2} (\rho_1 + \rho_2) (\rho_2 + \rho_3) (\rho_3 + \rho_1) (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1)^{-\frac{1}{2}}.$$

Solution by E. MORGAN, M.A.; R. KNOWLES, B.A.; and others.

If Δ be the area of a triangle, R radius of circumscribed circle, 2σ perimeter of orthocentric triangle; then, by a known theorem, $\Delta = R\sigma$. Now, in any triangle,

$$r_1 + r_2 + r_3 = r + 4R,$$

$$r = r_1 r_2 r_3 + (r_1 r_2 + r_2 r_3 + r_3 r_1), \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = (\text{semi-perimeter})^2.$$

Hence, for the orthocentric triangle, we have

$$\sigma = (\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1)^{\frac{1}{2}},$$

$$\text{and } 2R = \rho_1 + \rho_2 + \rho_3 - \rho_1 \rho_2 \rho_3 + \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1$$

$$= (\rho_1 + \rho_2) (\rho_2 + \rho_3) (\rho_3 + \rho_1) + \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1, \text{ therefore, \&c.}$$

[This, of course, is the same as proving that the area of the triangle formed by the escribed centres of any triangle is, with the usual notation, $\frac{1}{2} (r_1 + r_2)(r_2 + r_3)(r_3 + r_1)(r_1 r_2 + r_2 r_3 + r_3 r_1)^{-\frac{1}{2}}$.]

6791. (By E. W. SYMONS, M.A.)—A chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{(a^2 - b^2)^2}$; prove that the normals at its extremities meet on the ellipse.

Solution by D. EDWARDS; E. RUTTER; and others.

The normals at the ends of the chords

$$\frac{lx}{a} + \frac{my}{b} = 1, \quad \frac{x}{al} + \frac{y}{bm} = -1,$$

meet in a point (WOLSTENHOLME'S *Problems*); hence, if one chord be $\frac{ax}{\cos \phi} - \frac{by}{\sin \phi} = a^2 - b^2$, the other will be $\frac{x \cos \phi}{a^3} - \frac{y \sin \phi}{b^3} = -\frac{1}{a^2 - b^2}$ and the envelope of this is $\frac{x^2}{a^6} + \frac{y^2}{b^6} = \frac{1}{(a^2 - b^2)^2}$.

6625. (By C. LEUBESDORF, M.A.)—Equilateral triangles BDC, CEA, AFB are drawn externally on the sides of a plane triangle ABC; prove that, if AD, BE be joined, they will intersect at a point whose distance from AB is $\frac{4}{3} \cdot \frac{(AFC)(BFC)}{(ABC) + (BCD) + (CAE) + (ABF)}$, where (AFC) denotes the area of the triangle AFC, &c.

Solution by the Rev. T. R. TERRY, M.A., F.R.A.S.; E. RUTTER; and others.

If $l = (CAF) = (BAE)$, &c., then $(BCD) = m + n - \Delta$. Equation to AD is $my = nx$, and to BE is $lx = nz$. Hence AD, BE, CF meet in the point where $lx = my = nz \equiv \frac{\Delta lmn}{lm + mn + nl}$; therefore perpendicular from the point on AB = $\frac{2\Delta}{AB} \cdot \frac{lm}{lm + mn + nl}$. To throw this result into the form given in question, we must prove that

$$2(lm + mn + nl) = \Delta(l + m + n) \dots \dots \dots (1).$$

Now, since $m + n - \Delta = pa^2$, &c., where $p = \frac{1}{3}\sqrt{3}$, $2l = \Delta + p(b^2 + c^2 - a^2)$, therefore $4(lm + mn + nl) = 2\Delta[3\Delta + p(a^2 + b^2 + c^2)] = 2\Delta(l + m + n)$, the required result.

6644. (By W. J. C. SHARP, M.A.)—When the catalecticant of a binary $2n$ -ic $(a, b, c \dots k, l)(x, y)^{2n}$ vanishes, prove that the n quantities to the sum of whose $2n^{\text{th}}$ powers the quantic is reducible are the factors of the canonizant of $(a, b, c \dots k)(x, y)^{2n-1}$.

Solution by the PROPOSER.

Let $p_1x + q_1y$, $p_2x + q_2y$, &c. be the n quantities required,

$$\therefore (a, b, c \dots k, l) (x, y)^{2n} \equiv (p_1x + q_1y)^{2n} + (p_2x + q_2y)^{2n} + \&c.,$$

$$\therefore ax + by \equiv p_1^{2n-1} (p_1x + q_1y) + p_2^{2n-1} (p_2x + q_2y) + \&c.,$$

$$\begin{array}{ccc} \delta x + cy \equiv p_1^{2n-2} q_1 (p_1x + q_1y) + p_2^{2n-2} q_2 (p_2x + q_2y) + \&c., \\ \&c. & \&c. & \&c. \end{array}$$

Hence, by substitution,

$$\left| \begin{array}{ccc} ax + by & bx + cy & cx + dy \dots \\ \delta x + cy & cx + dy & dx + by \dots \\ \&c. & \&c. & \&c. \end{array} \right| \begin{array}{l} \text{vanishes if } p_1x + q_1y = 0, \\ \text{or if } p_2x + q_2y = 0, \&c. \end{array}$$

and therefore $p_1x + q_1y$, &c. are the factors, &c.

6769. (By Professor TOWNSEND, F.R.S.)—Three forces in a common space being supposed transferred to the centre of the quadric determined by their three lines of direction in the space; show, on elementary principles, that

(a) The plane of the resultant moment is conjugate to the direction of the resultant force with respect to the surface;

(b) The pitch of the equivalent wrench = $abc + r^2$; where a, b, c are the semi-axes of the surface, and r its radius in the direction of the force.

Solution by the PROPOSER; T. WOODCOCK, B.A.; and others.

Of these properties, which are true generally for any number of forces acting along generators of the same system of any ruled quadric in their space, the first, which has already been given separately in Quest. 6535, is at once manifest from the consideration that as the forces, whatever be their number, have no moments round the two generators E and F of the opposite system at the two extremities P and Q of the diameter POQ, which coincides in direction with the resultant of their transference to the centre O of the surface, and as the planes of their moments at P and Q are parallel to each other and to that of their moment at O, the latter is in consequence parallel to the two tangent planes to the surface at P and Q, and is therefore the diametral plane conjugate to the direction PQ; and the second may be readily proved from it as follows.

Denoting by x, y, z the three coordinates to the three principal planes of the surface of either extremity P of the diameter PQ, by r and p the radius OP and the perpendicular from O on the tangent plane at P, by a, b, c the three semi-axes of the surface, by X, Y, Z the three components parallel to the axes of the resultant R of the system of forces transferred to O, and by L, M, N those round the same of the corresponding moment (1) resulting from the transference; then, since, by the above,

$$L : M : N = \frac{X}{a^2} : \frac{Y}{b^2} : \frac{Z}{c^2} = \frac{x}{a^2} \cdot \frac{R}{r} : \frac{y}{b^2} \cdot \frac{R}{r} : \frac{z}{c^2} \cdot \frac{R}{r},$$

$$\text{therefore } L = k \cdot \frac{X}{a^2} = k \cdot \frac{R}{r} \cdot \frac{x}{a^2}, \quad M = k \cdot \frac{R}{r} \cdot \frac{y}{b^2}, \quad N = k \cdot \frac{R}{r} \cdot \frac{z}{c^2},$$

where k remains to be determined; hence

$$G = [L^2 + M^2 + N^2]^{\frac{1}{2}} = k \cdot \frac{R}{r} \cdot \left[\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right]^{\frac{1}{2}} = \frac{k}{rp} R,$$

and therefore $\frac{G}{R} = \frac{k}{rp}$, and $\frac{G}{R} \cdot \frac{p}{r} = \text{pitch of wrench} = \frac{k}{r^2}$;

and it remains only to show that the value of k is that given above.

Conceiving the entire system of forces transformed, as they evidently may, into two pairs of forces P and P' , Q and Q' acting along the two pairs of generators E and E' , F and F' of their system passing through the two pairs of opposite extremities A and A' , B and B' of the two real axes AOA' and BOB' of the surface; we have then, evidently,

$$X = (Q \mp Q') \frac{a}{(a^2 + c^2)^{\frac{1}{2}}}, \quad Y = (P \mp P') \frac{b}{(b^2 + c^2)^{\frac{1}{2}}},$$

$$Z = (P \pm P') \frac{c}{(a^2 + c^2)^{\frac{1}{2}}} + (Q \pm Q') \frac{c}{(b^2 + c^2)^{\frac{1}{2}}}, \quad L = (Q \mp Q') \frac{bc}{(a^2 + c^2)^{\frac{1}{2}}},$$

$$M = (P \mp P') \frac{ac}{(b^2 + c^2)^{\frac{1}{2}}}, \quad N = (P \pm P') \frac{ab}{(a^2 + c^2)^{\frac{1}{2}}} + (Q \pm Q') \frac{ab}{(b^2 + c^2)^{\frac{1}{2}}},$$

and consequently

$$\frac{L}{X} = \frac{bc}{a} = \frac{abc}{a^2}, \quad \frac{M}{Y} = \frac{ac}{b} = \frac{abc}{b^2}, \quad \frac{N}{Z} = \frac{ab}{c} = \frac{abc}{c^2};$$

hence $k = abc$, and therefore, &c.

6962. (By J. J. WALKER, M.A.)—Show that the cosine of half the angle between tangents drawn from a point on an ellipse to an inner confocal ellipse, varies inversely as the diameter of the former parallel to its tangent at that point.

7160. (By J. GRIFFITHS, M.A.)—If 2α be the angle between the pair of tangents from an external point (x', y') to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and a_1, b_1 the semi-axes of the confocal ellipse through (x', y') , show that

$$\sin^2 \alpha = \frac{a_1^2 a^2 y'^2 + b_1^2 b^2 x'^2}{a_1^4 y'^2 + b_1^4 x'^2}.$$

Solution by J. J. WALKER, M.A.; W. M. MEE, B.A.; and others.

We have $\tan^2 2\alpha = \frac{4(b^2 x'^2 + a^2 y'^2 - a^2 b^2)}{[x'^2 + y'^2 - (a^2 + b^2)]^2}$, or, if $k^2 = a_1^2 - a^2 = b_1^2 - b^2$,

r_1 = semi-diameter of outer ellipse parallel to its tangent at $x'y'$,

$$\tan^2 2\alpha = \frac{4k^2(r_1^2 - k^2)}{(2k^2 - r_1^2)^2};$$

whence $(2 \cos^2 \alpha - 1)^2 = \frac{1}{1 + \tan^2 2\alpha} = \frac{(2k^2 - r_1^2)^2}{r_1^4}, \cos \alpha = \frac{k}{r_1}.$

$$\text{Again, } \sin^2 \alpha = 1 - \cos^2 \alpha = \frac{r_1^2 - k^2}{r_1^2} = \frac{a_1^4 y^2 + b_1^4 x^2 - k^2 a_1^2 b_1^2}{a_1^4 y^2 + b_1^4 x^2} \\ = \frac{a_1^4 y^2 + b_1^4 x^2 - (a_1^2 - a^2) a_1^2 y^2 - (b_1^2 - b^2) b_1^2 x^2}{a_1^4 y^2 + b_1^4 x^2} = \frac{a_1^2 a^2 y^2 + b_1^2 b^2 x^2}{a_1^4 y^2 + b_1^4 x^2}.$$

6623. (By Professor HUDSON, M.A.)—If the angular radius of the small circle drawn round a triangle be $\tan^{-1} 2$, the area of the triangle be one-sixth of the sphere, and one side be a quadrant, find all the parts of the triangle.

Solution by W. M. COATES; CHARLOTTE A. SCOTT; and others.

We have $\tan R = \frac{\tan \frac{1}{2}a}{\cos(S-A)}$, since $a = \frac{1}{2}\pi$ and $R = \tan^{-1} 2$;

therefore $2 = \sec(S-A)$, therefore $S-A = \frac{1}{2}\pi$ or $B+C-A = \frac{3}{2}\pi$.

But $A+B+C-\pi = \frac{3}{2}\pi$, therefore $A = \frac{1}{2}\pi$,

and we have two equations to determine B and C , namely $B+C = \frac{3}{2}\pi$, $\cos A = 0 = \cot B \cot C$, therefore either B or $C = \frac{1}{2}\pi$. If $B = \frac{1}{2}\pi$, we have

$$a = \frac{1}{2}\pi, \quad b = \frac{1}{2}\pi, \quad c = \frac{3}{2}\pi, \quad A = \frac{1}{2}\pi, \quad B = \frac{1}{2}\pi, \quad C = \frac{3}{2}\pi.$$

The triangle is therefore that included on a globe between the equator and the meridians of 0° and 120° .

6159. (By Professor COCHEZ.)—Un paraboloïde elliptique est coupé par un plan perpendiculaire à l'axe. Trouver le plus grand parallépipède rectangle que l'on puisse inscrire dans la portion de paraboloïde déterminée par le plan sécant.

Solution by Rev. J. L. KITCHIN, M.A.; SARAH MARKS; and others.

Let $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ be the paraboloid, $z = h$ the cutting plane, and the four angular points in the parallel plane, the angular points of a rectangle inscribed in an ellipse, be $(x_1, y_1, z_1)(x_1, -y_1, z_1), (-x_1, y_1, z_1)(-x_1, -y_1, z_1)$. Then the volume is $4x_1y_1(h-z_1)$;

therefore $x_1y_1 \left\{ h - \frac{1}{2} \left(\frac{x_1^2}{a} + \frac{y_1^2}{b} \right) \right\} = \text{a maximum,}$

$$\therefore y_1 \left\{ h - \frac{1}{2} \left(\frac{x_1^2}{a} + \frac{y_1^2}{b} \right) \right\} - \frac{x_1^2 y_1}{a} = 0, \quad x_1 \left\{ h - \frac{1}{2} \left(\frac{x_1^2}{a} + \frac{y_1^2}{b} \right) \right\} - \frac{x_1 y_1^2}{b} = 0;$$

therefore $\frac{x_1^3 y_1}{a} - \frac{y_1^3 x_1}{b} = 0$, therefore $x_1 = 0, y_1 = 0$, which gives an evident

minimum, or $\frac{x_1^3}{a} - \frac{y_1^3}{b} = 0$; therefore $\frac{x_1^3}{a} = \frac{y_1^3}{b} = \frac{h}{2} = z_1$, and the

$$\text{volume} = h^3 (ab)^{\frac{1}{3}}.$$

7141. (By R. TUCKER, M.A.)—Any fourth point (P) is taken on the circumference of the circle ABC; prove that the mid-points of PA, PB, PC form a triangle, similar to ABC and of one quarter its area, and such that its circumscribing circle always touches ABCP at P. [Further results may be obtained by substituting any one of the A, B, C points for P.]

Solution by Rev. T. R. TERRY, M.A.; J. O'REGAN; and others.

If a be the middle point of PA, and PT be the tangent at P to the circle ABC, then clearly ab is parallel to AB and half of it, therefore $\triangle abc$ is similar to $\triangle ABC$ and one quarter its area. Again, since $\angle APB = \angle ACB = \angle acb$, therefore a circle goes through $abcP$; and since $\angle APT = \angle ABP = \angle abP$, the circles ABCP, $abcP$ touch at P.

7133. (By Rev. G. RICHARDSON, M.A.)—If AOD, BOE, COF be the perpendiculars from A, B, C on the opposite sides of the triangle ABC, and G be the centroid of the triangle; prove that the circles described about the triangles ADG, BEG, CFG all meet again in a point which is the intersection of OG with the radical axis of the nine-point circle, and circumscribing circle of ABC.

Solution by Prof. GENESE, M.A.; M. JENKINS, M.A.; and others.

1. Since $AO \cdot OD = BO \cdot OE = CO \cdot OF$, the line GO meets the three circles in the same point, —K say.

2. $GO \cdot OK = AO \cdot OD = \frac{1}{4}(R^2 - SO^2)$. S being the centre of the circle ABC, therefore $OK = \frac{1}{4} \cdot \frac{R^2 - SO^2}{SO}$. Let N be the centre of the

N. P. C, M the middle point of SN, then $MO = \frac{3}{4}OS$, $\therefore MK = \frac{1}{4} \frac{R^2}{SO}$, or $2SN \cdot MK = \frac{3}{4}R^2 = R^2 - (\frac{1}{4}R)^2$, therefore K is on the radical axis of circle ABC and N. P. C.

6842. (By G. F. WALKER, M.A.)—A system of $2n$ convex lenses, of equal numerical focal length f are placed with their axes in the same straight line and their centres at a distance $4f$ apart, except the two middle ones, which are at a distance $8f$ apart; show that the focal length of a lens which must be placed midway between the two middle ones in order that the image of a bright point at a distance $4f$ in front of the first lens may be formed at an equal distance behind the last lens, is $\frac{2(n+1)}{2n+1}f$.

Solution by G. M. REEVES, M.A.; BELLE EASTON; and others.

Let $v_1, v_2, v_3 \dots v_n$ be the distances from each of the n lenses, on one side, of the images after refraction through that lens; then we have the n equations

$$\frac{1}{v_1} - \frac{1}{4f} = -\frac{1}{f}, \quad \frac{1}{v_2} - \frac{1}{v_1 + 4f} = -\frac{1}{f}, \quad \dots \quad \frac{1}{v_n} - \frac{1}{v_{n-1} + 4f} = -\frac{1}{f},$$

whence
$$\frac{1}{v_1} = -\frac{3}{4f}, \quad \frac{1}{v_2} = -\frac{5}{8f}, \quad \frac{1}{v_3} = -\frac{7}{12f};$$

and, by mathematical induction, we can prove that $\frac{1}{v_n} = -\frac{2n+1}{4nf}$. Let

F be the numerical focal length of the required lens; then, on account of the symmetrical arrangement of the $2n$ lenses,

$$-\frac{1}{v_n + 4f} - \frac{1}{v_n + 4f} = -\frac{1}{F}, \quad \text{or} \quad \frac{1}{F} = \frac{2}{v_n + 4f} = \frac{2}{4f - \frac{4nf}{2n+1}} = \frac{2n+1}{2(n+1)f},$$

therefore

$$F = \frac{2(n+1)}{2n+1} f.$$

6734. (By H. G. DAWSON.)—If $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ have a double root a , prove that $H = b^2 - ac$, $I = ae - 4bd + 3c^2$,

$$J = ace + 2bcd - ad^2 - eb^2 - c^3, \quad a = -\frac{b}{a} + \left(\frac{H}{a^2} \pm \frac{3J}{aI}\right)^{\frac{1}{2}}.$$

Solution by SARAH MARKS; G. EASTWOOD, M.A.; and others.

Transforming $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ by substituting $y + a = x$, the equation becomes

$$Ay^4 + 4By^3 + 6Cy^2 = 0, \quad \frac{H}{a^2} = \frac{B^2 - AC}{A^2}, \quad \frac{I}{a^2} = \frac{3C^2}{A^2}, \quad \text{therefore} \quad \left(\frac{B}{A}\right)^2 = \frac{H}{a^2} + \left(\frac{I}{3a^2}\right),$$

therefore
$$\left(\frac{B}{A}\right)^2 = \frac{H}{a^2} \pm \frac{3J}{aI};$$

but
$$\beta - a + \gamma - a = -\frac{4B}{A}, \quad \text{therefore} \quad -\frac{4b}{a} - 4a = -\frac{4B}{A},$$

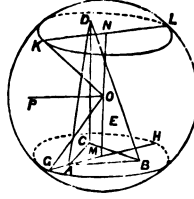
therefore
$$a = -\frac{b}{a} + \frac{B}{A}, \quad \text{therefore, \&c.}$$

6164. (By Professor SEITZ, M.A.)—If A, B, C, D, E are five random points within a sphere, prove that the chance that any one of the points is within the tetrahedron having the other four points for its vertices is $\frac{2}{125}$.

6169. (By ELIZABETH BLACKWOOD.)—If A, B, C are random points within a sphere of which O is the centre and r the radius, prove that the average volume of the tetrahedron $ABCO$ is $\frac{2}{105\pi} r^3$.

Solution by Prof. SEITZ, M.A.; Prof. MATZ, M.A.; and others.

Let GH be the diameter of the section of the sphere made by the plane through A, B, C, M its centre, GL the diameter of the section through D parallel to the plane ABC, N its centre, O the centre of the sphere, OP a line such that AB is parallel to the plane MOP.



Now, the required chance is evidently equal to five times the chance that E is within the tetrahedron ABCD.

Let $OG=r$, $\angle GOM=\theta$, and $\angle KON=\mu$: then, giving D all positions within the sphere, we have, for the sum of all the tetrahedrons ABCD,

$$\begin{aligned} & \int_0^{\pi-\theta} \text{area ABC} \times \frac{1}{3} (\cos \mu + \cos \theta) \pi r^4 \sin^3 \mu \, d\mu \\ & + \int_0^{\theta} \text{area ABC} \frac{1}{3} \times (\cos \mu - \cos \theta) \pi r^4 \sin^3 \mu \, d\mu \\ & = \frac{1}{15} \pi r^4 (8 - 4 \sin^2 \theta - \sin^4 \theta) \text{area ABC} \dots \dots (1). \end{aligned}$$

Now, let $MA=x$, $AB=y$, $AC=z$, $\angle BAM=\phi$, $\angle CAM=\psi$, $\angle MOP=\omega$, and the angle the plane MOP makes with a fixed plane through $OP=\rho$. Then $(1) = \frac{1}{15} (8 - 4 \sin^2 \theta - \sin^4 \theta) \pi r^4 yz \sin(\phi + \psi) = f(\theta) yz \sin(\phi + \psi)$. An element of the sphere at A is $r \sin \theta \, d\theta \cdot 2\pi x \, dx$, at B it is $y^2 dy \, d\phi \, d\omega$, and at C it is $\sin(\phi + \psi) \sin \omega x^2 dx \, d\psi \, d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x , 0 and $r \sin \theta = x$, and tripled; of ϕ , $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ , $-\phi$ and $\frac{1}{2}\pi$, and doubled; of ω , 0 and π ; of ρ , 0 and 2π ; of y , 0 and $2x \cos \phi = y'$; and of z , 0 and $2x \cos \psi = z'$. Hence, since the whole number of ways the five points can be taken is $(\frac{1}{2}\pi r^3)^5$, the chance that E is within the tetrahedron ABCD is

$$\begin{aligned} & \frac{6}{(\frac{1}{2}\pi r^3)^5} \int_0^{\frac{1}{2}\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi} \int_0^{\pi} \int_0^{2\pi} \int_0^{y'} f(\theta) r \sin \theta \, d\theta \cdot 2\pi x \, dx \, d\phi \sin^2(\phi + \psi) \, d\psi \\ & \quad \times \sin \omega \, d\omega \, d\rho \, y^3 \, dy \, x^3 \, dx \\ & = \frac{729}{16\pi^4 r^{14}} \int_0^{\frac{1}{2}\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi} \int_0^{\pi} \int_0^{2\pi} f(\theta) \sin \theta \, d\theta \, x^2 \, dx \cos^4 \phi \, d\phi \sin^2(\phi + \psi) \\ & \quad \cos^4 \psi \, d\psi \sin \omega \, d\omega \, d\rho \\ & = \frac{729}{4\pi^3 r^{14}} \int_0^{\frac{1}{2}\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi} f(\theta) \sin \theta \, d\theta \, x^2 \, dx \cos^4 \phi \, d\phi \sin^2(\phi + \psi) \cos^4 \psi \, d\psi \\ & = \frac{243}{64\pi^2 r^{14}} \int_0^{\frac{1}{2}\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\theta) [3(\frac{1}{2}\pi + \phi)(5 - 4 \cos^2 \phi) + 15 \sin \phi \cos \phi \\ & \quad - 2 \sin \phi \cos^3 \phi] \sin \theta \, d\theta \, x^2 \, dx \cos^4 \phi \, d\phi \\ & = \frac{405}{4096 r^{10}} \int_0^{\frac{1}{2}\pi} \int_0^{x'} (8 - 4 \sin^2 \theta - \sin^4 \theta) \sin \theta \, d\theta \, x^2 \, dx \\ & = \frac{81}{8192} \int_0^{\frac{1}{2}\pi} (8 - 4 \sin^2 \theta - \sin^4 \theta) \sin^{11} \theta \, d\theta = \frac{9}{716}. \end{aligned}$$

Hence the required chance is $\frac{9}{143}$.

7069. (By D. EDWARDS.)—If x, y, z be the distances of a point P from the angular points of a triangle, prove that the mean value of $x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C$, as P ranges over the circle about ABC, is three times the area of the triangle.

Solution by W. H. BLYTHE, M.A. ; W. M. COATES, B.A. ; and others.

The expression given has by a former question been shown $= 4 (R^2 + d^2) \sin A \sin B \sin C$, where d is the distance of P from the centre of the circle round ABC. The mean value required

$$= \int_0^R 4 (R^2 + x^2) \sin A \sin B \sin C \cdot 2\pi x \cdot dx + \int_0^R 2\pi x \cdot dx \\ = 6R^2 \sin A \sin B \sin C.$$

Putting $R \sin A = \frac{1}{2}a$, $R \sin B = \frac{1}{2}b$, this evidently reduces to three times the area of the triangle.

7093. (By H. L. ORCHARD, M.A.)—If C, S, respectively, be the centre and one focus of the ellipse $3(x^2 - a^2) + 4y^2 = 0$, and P be a point on the curve such that the $\angle CSP = \theta$; find (1) the area of the triangle CSP, and (2) for what value of θ this will be a maximum.

Solution by Rev. J. L. KITCHIN, M.A. ; C. MORGAN, B.A. ; and others.

The semi-axes of the ellipse are a and $\frac{1}{2}a\sqrt{3}$, therefore $e = \frac{1}{2}$; also area of CSP $= \frac{1}{2}ac \cdot SP \cdot \sin \theta = \frac{1}{2}ac \frac{a(1-e^2)}{1-e \cos \theta} \sin \theta = \frac{3}{16}a^2 \cdot \frac{\sin \theta}{1-\frac{1}{2} \cos \theta}$. For a maximum the difficulty with respect to θ vanishes, whence we obtain $\theta = 60^\circ$.

6975. (By E. RUTTER.)—If $s_1 =$ sum of odd terms, and $s_2 =$ sum of even terms, of $(a+b)^n$, show that $s_1^2 - s_2^2 = (a^2 - b^2)^n$.

Solution by H. L. ORCHARD, M.A. ; J. L. JENKINS, M.A. ; and others.

$$s_1^2 - s_2^2 = (s_1 + s_2)(s_1 - s_2) = (a+b)^n (a-b)^n = (a^2 - b^2)^n.$$

6968. (By D. EDWARDS.)—Prove that
 $(y+z-2x')(x+x'-2y')(x+y-2x') - (y+z-2x')(x+x'-y'-x'')^2 - \dots - \dots$
 $\equiv 2(x+x'-y'-x'')(y+y'-x'-x'')(x+z'-x'-y').$

Solution by G. HEPPLE, M.A.; SARAH MARKS; and others.

Putting $u = x + x' - y' - z'$; $v = &c.$; $w = &c.$; the identity becomes

$$(v + w)(w + u)(u + v) - (v + w)u^2 - (w + u)v^2 - (u + v)w^2 = 2uvw,$$

which is obviously true.

7021. (By W. NICHOLLS, B.A.)—Let ABCD be any quadrilateral circumscribable by a circle, P the intersection of the sides AD, BC; L the middle point of CD; and M the point where PL produced meets AB: prove that

$$MA : MB = AP^2 : PB^2.$$

Solution by J. S. JENKINS; J. O'REGAN; and others.

$$\begin{aligned} \frac{MA}{MB} &= \frac{MA}{MP} \cdot \frac{MP}{MB} = \frac{\sin LPD}{\sin A} \cdot \frac{\sin B}{\sin LPC} = \frac{\sin LPD}{\sin L} \cdot \frac{\sin L}{\sin LPC} \cdot \frac{\sin B}{\sin A} \\ &= \frac{LD}{PD} \cdot \frac{PC}{LO} \cdot \frac{\sin D}{\sin C} = \frac{PC}{PD} \cdot \frac{PC}{PD}. \end{aligned}$$

5615. (By S. TEBAY, B.A.)—"In the arbelon of the ancient geometers, three circular segments are drawn on the same straight line (length 6), and on given parts of it (4 and 2)." Given the sum of the three arcs, find the radii when the sum of the areas of the segments is a maximum.

Solution by the PROPOSER; A. MARTIN, M.A.; and others.

Let $2a_1, 2a_2, \dots$ be the chords, $2\theta_1, 2\theta_2, \dots$ the angles which they subtend at the centres, $r_1, r_2, \dots$ the radii, and $2s$ the sum of the arcs. Then

$$s = r_1\theta_1 + r_2\theta_2 + \dots = a_1\theta_1 \operatorname{cosec} \theta_1 + a_2\theta_2 \operatorname{cosec} \theta_2 + \dots,$$

$$r_1^2\theta_1 + r_2^2\theta_2 + \dots = (a_1r_1 \cos \theta_1 + a_2r_2 \cos \theta_2 + \dots)$$

$$= a_1^2\theta_1 \operatorname{cosec}^2 \theta_1 + a_2^2\theta_2 \operatorname{cosec}^2 \theta_2 + \dots - (a_1^2 \cot \theta_1 + a_2^2 \cot \theta_2 + \dots),$$

a maximum. Differentiating, we have

$$a_1 \operatorname{cosec} \theta_1 (1 - \theta_1 \cot \theta_1) \delta\theta_1 + a_2 \operatorname{cosec} \theta_2 (1 - \theta_2 \cot \theta_2) \delta\theta_2 + \dots = 0,$$

$$a_1^2 \operatorname{cosec}^2 \theta_1 (1 - \theta_1 \cot \theta_1) \delta\theta_1 + a_2^2 \operatorname{cosec}^2 \theta_2 (1 - \theta_2 \cot \theta_2) \delta\theta_2 + \dots = 0.$$

Multiply the first by λ , and add; then, putting the coefficient of $\delta\theta_1$,

$$\delta\theta_2, \dots = 0, \text{ we have } \lambda + a_1 \operatorname{cosec} \theta_1 = 0, \quad \lambda + a_2 \operatorname{cosec} \theta_2 = 0, \text{ \&c.,}$$

or

$$-\lambda = r_1 = r_2 = \text{\&c.} = r, \text{ suppose;}$$

$$\text{therefore} \quad \frac{s}{r} = \theta_1 + \theta_2 + \dots = \sin^{-1} \frac{a_1}{r} + \sin^{-1} \frac{a_2}{r} + \dots;$$

which determines r . In the particular case,

$$\frac{6}{r} = \sin^{-1} \frac{1}{r} + \sin^{-1} \frac{2}{r} + \sin^{-1} \frac{3}{r};$$

from which we find $r = 23.005$, nearly.

6761. (By G. F. WALKER, M.A.)—A railway engine is drawing a train of equal carriages connected by spring couplings of strength μ , and the driving power is so adjusted that the velocity is $a + b \sin nt$. Show that, if n be nearly equal to $b \left\{ \frac{2\mu}{(M + 4m)b^2 + 4m\kappa^2} \right\}^{\frac{1}{2}}$, the couplings will probably break, M being the mass of a carriage, which is supported on four equal wheels of mass m , radius b , and radius of gyration κ .

Solution by the PROPOSER; E. RUTTER; and others.

Let y_r be the distance of the centre of the r^{th} carriage from some fixed point on the line.

Then, the coupling chains being supposed inextensible, and their mass being neglected, the equation of motion of the r^{th} carriage is found to be

$$M \ddot{y}_r + 4m \ddot{y}_r + 4m\kappa^2 \frac{\ddot{y}_r}{b} = \frac{y_{r+1} - 2y_r + y_{r-1}}{2} \mu,$$

or
$$\frac{4}{n_1^2} \ddot{y}_r = y_{r+1} - 2y_r + y_{r-1},$$

where n_1 has the value given in the question. We shall now find the period of the small oscillation of the strain without the engine.

The first and last equations of motion

$$\frac{4}{n_1^2} \ddot{y}_1 = y_2 - y_1, \quad \frac{4}{n_1^2} \ddot{y}_s = -(y_s - y_{s-1}),$$

if s be the number of carriages.

Assume $y_r = z_r \sin pt$, and we find

$$z_{r+1} - \left(1 - \frac{2p^2}{n_1^2}\right) z_r + z_{r-1} = 0, \quad z_s - \left(1 - \frac{4p^2}{n_1^2}\right) z_1 = 0,$$

$$z_s \left(1 - \frac{4p^2}{n_1^2}\right) - z_{s-1} = 0.$$

Let a_1, a_2 be the roots of the equation $x^2 - 2 \left(1 - \frac{2p^2}{n_1^2}\right) x + 1 = 0$.

Then $z_r = Aa_1^r + Ba_2^r$ and $a_1 + a_2 = 2 \left(1 - \frac{2p^2}{n_1^2}\right)$, $a_1 a_2 = 1$,

$$Aa_1^2 + Ba_2^2 - (a_1 + a_2 - 1)(Aa_1 + Ba_2) = 0,$$

$$(Aa_1^s + Ba_2^s)(a_1 + a_2 - 1) - (Aa_1^{s-1} + Ba_2^{s-1}) = 0$$

or, remembering $a_1 a_2 = 1$, $A(a_1 - 1) + B(a_2 - 1) = 0$,

and $Aa_1^s(a_1 - 1) + Ba_2^s(a_2 - 1) = 0$, $\therefore a_1^s = a_2^s$.

But, if $1 - \frac{2p^2}{n_1^2}$ be numerically greater than unity, the roots a_1 and a_2 are real and different; hence it must be less than unity, = $\cos \theta$ suppose,

$$1 - \frac{2p^2}{n_1^2} = \cos \theta, \text{ therefore } p = n_1 \sin \frac{\theta}{2}.$$

Also $a_1 = \cos \theta + \sqrt{-1} \sin \theta$, $a_2 = \cos \theta - \sqrt{-1} \sin \theta$, and if $a_1^s = a_2^s$ we have $\sin s\theta = 0$, therefore $s\theta = \kappa\pi$, $\theta = \frac{\kappa\pi}{s}$, therefore $p = n_1 \sin \frac{\kappa\pi}{2s}$.

This therefore gives the period of the natural oscillation, and if the period of the driving power nearly coincide with any one of these, large relative motion will probably ensue, and the couplings probably break. κ has here an integral value, and $p = n_1$, or $\kappa = s$ is the only one which gives a result independent of the number of carriages.

Only the oscillatory part of the motion is above taken into account, and the b in $a + b \sin nt$, which had better be written $a + \beta \sin nt$, has no connection with the radius of the wheels.

6604. (By Professor MATZ, M.A.)—Prove that the mean area of the maximum elliptic disc placed upon the surface common to two equal circles of radius unity, that are thrown upon each other at random, is $\frac{1}{2}\pi(3\pi - 8)$.

Solution by Professor SEITZ, M.A.

Let A and B be the centres of the two circles, CD the common chord of the circles, and OM, ON the semi-axes of the maximum elliptic disc. Let OM = a , ON = b , and AB = z .

Then the equation to the circle A, referred to AB, CD, is $(x + \frac{1}{2}z)^2 + y^2 = 1$(1); and the equation to the ellipse is

$$a^2x^2 + b^2y^2 = a^2b^2$$
.....(2).

From (1) and (2), by eliminating y , we have

$$(a^2 - b^2)x^2 - b^2zx + b^2(1 - a^2 - \frac{1}{4}z^2) = 0$$
.....(3).

But, since the ellipse is tangent to the circle, the two values of x in (3) must be equal; therefore, $(a^2 - b^2)(4 - 4a^2 - z^2) = b^2z^2$(4).

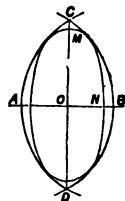
From (4) we find $a^2b^2 = a^4 - \frac{a^4z^2}{4(1 - a^2)} = \text{a maximum}$(5).

Differentiating (5), and reducing, we find

$$(2 - a^2)z^2 = 8(1 - a^2)^2$$
.....(6);

and from (6) we find $z^2 = \frac{8(1 - a^2)^2}{2 - a^2}$(7).

The value of z in (5) gives $ab = \frac{a^3}{(2 - a^2)^{\frac{1}{2}}}$(8).



Differentiating (7), we have $z dz = \frac{8a da}{(2-a^2)^2} - 8a da \dots\dots\dots(9)$.

Making use of (8) and (9), and observing that, when $z = 0$, $a = 1$, and when $z = 2$, $a = 0$, we have, for the mean value of the elliptic disc,

$$\begin{aligned}\Delta &= \int_0^2 \pi ab \cdot 2\pi z dz + \int_0^2 2\pi z dz = \frac{1}{2}\pi \int_0^2 ab z dz \\ &= 4\pi \int_0^1 \frac{a^4 da}{(2-a^2)^{\frac{3}{2}}} - 4\pi \int_0^1 \frac{a^4 da}{(2-a^2)^{\frac{3}{2}}}.\end{aligned}$$

$$\text{Let } a = \sqrt{2} \sin \theta; \text{ then } \Delta = 16\pi \int_0^{\frac{1}{2}\pi} \sin^4 \theta d\theta - 4\pi \int_0^{\frac{1}{2}\pi} \tan^4 \theta d\theta = \frac{1}{2}\pi (3\pi - 8).$$

6798. (By Professor WOLSTENHOLME, M.A.)—If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on an ellipse ($b^2x^2 + a^2y^2 = a^2b^2$), prove that (1) the area of the triangle whose corners are the vertices of the quadrangle is

$$\pm 4ab \frac{\sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\alpha - \delta) \sin \frac{1}{2}(\gamma - \alpha) \sin \frac{1}{2}(\beta - \delta) \sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta)}{\sin \frac{1}{2}(\beta + \gamma - \alpha - \delta) \sin \frac{1}{2}(\gamma + \alpha - \beta - \delta) \sin \frac{1}{2}(\alpha + \beta - \gamma - \delta)};$$

(2) this is also the area of the triangle formed by the diagonals of the quadrilateral formed by the tangents at the four points, the two triangles being identical; and (3) if the normals at the four points meet in one point (X, Y), the area of the triangle is

$$ab \frac{\{[a^2X^2 + b^2Y^2 - (a^2 - b^2)^2] + 27a^2b^2(a^2 - b^2)^2 X^2Y^2\}^{\frac{1}{2}}}{(a^2 - b^2)(a^2X^2 - b^2Y^2)}.$$

Solution by D. EDWARDS; BELLE EASTON; and others.

Substituting the values of the coordinates of the vertices in the expression in Cartesian coordinates for the area, we have $\Delta + ab$ equal to a fraction whose numerator is

$$\sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta) \sin^2 \frac{1}{2}(\alpha + \beta - \gamma - \delta) + \dots + \dots,$$

and denominator as in question. This denominator

$$= \pm (\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta),$$

and from the biquadratics in $\cos \theta$ or $\sin \theta$ we see that this

$$= \pm e^{-4}(a^2X^2 - b^2Y^2) \dots\dots\dots(1).$$

Also, since $\sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta) + \dots + \dots$ vanishes, writing

$$1 - \cos(\alpha + \beta - \gamma - \delta) \text{ for } 2 \sin^2 \frac{1}{2}(\alpha + \beta - \gamma - \delta), \text{ \&c.,}$$

then expanding the cosines, we get for the numerator of $4\Delta + ab$

$$\begin{aligned}&-8 \sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\gamma - \delta) \sin \frac{1}{2}(\alpha - \delta) \sin \frac{1}{2}(\beta - \delta) \sin \frac{1}{2}(\gamma - \delta) \\ &-8 \sin \frac{1}{2}(\gamma - \alpha) \sin \frac{1}{2}(\beta - \delta) \times \sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta) \cos \frac{1}{2}(\alpha - \beta) \cos \frac{1}{2}(\gamma - \delta) \\ &+ 2 \sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta) \cos(\alpha - \delta) \cos(\beta - \gamma) \\ &+ 2 \cos(\alpha - \beta) \cos(\gamma - \delta) \sin \frac{1}{2}(\beta - \alpha) \sin \frac{1}{2}(\gamma - \delta),\end{aligned}$$

$$\text{and} \quad \frac{1}{2} \left\{ 9 + \left(\frac{d\rho}{ds} \right)^2 \right\} + \frac{\left(\rho \frac{d^2\rho}{ds^2} \right)^2}{9 + \left(\frac{d\rho}{ds} \right)^2 - 3\rho \frac{d^2\rho}{ds^2}} = H^2,$$

where H is the absolute invariant $= \frac{2-e^2}{(1-e^2)^{3/2}}$, and e the excentricity.

7105. (By Professor MINCHIN, M.A.)—If (a, b, s) denote the elongations and shear with reference to any pair of rectangular axes at a point P in a natural solid subject to uniplanar strain, and if k_1, k_2 denote the radii of gyration, about the axes, of any small area, S , surrounding P in the plane of strain; show that the quantity $ak_2^2 + bk_1^2 - 2sp^2$ has the same value for all sets of axes at P , where Sp^2 denotes the product of inertia of S with respect to the axes.

Solution by G. EASTWOOD, M.A.; D. EDWARDS; and others.

Let K_1^2, K_2^2, P^2 refer to the new axes. Then

$$K_1^2 = k_1^2 \cos^2 \theta + k_2^2 \sin^2 \theta - p^2 \sin 2\theta, \quad K_2^2 = k_1^2 \sin^2 \theta + k_2^2 \cos^2 \theta + p^2 \sin 2\theta, \\ P^2 = (k_1^2 - k_2^2) \sin \theta \cos \theta + p^2 \cos 2\theta,$$

Also (MINCHIN'S *Statics*) $a' = a \cos^2 \theta + b \sin^2 \theta + s \sin 2\theta$,

$$b' = a \sin^2 \theta + b \cos^2 \theta - s \sin 2\theta, \quad s' = (b-a) \sin \theta \cos \theta + s \cos 2\theta.$$

Hence, substituting and reducing, we have

$$a'K_1^2 + b'K_2^2 - 2s'P^2 = ak_1^2 + bk_2^2 - 2sp^2.$$

7147. (By Professor HUDSON, M.A.)—An arc of a curve is in the shape of the portion of a catenary cut off by any horizontal base: find the law of density of the arc so that its centre of gravity may bisect the line drawn from the vertex to the middle point of the base.

Solution by Dr. CURTIS; W. J. O. SHARP, M.A.; and others.

In any curve symmetrical about a vertical axis Y , the centre of gravity of the curve intercepted between by any two horizontal lines will bisect the line joining the middle points of the intercepts on these lines, if the density ρ be proportional to $\frac{dy}{ds}$, that is, the sine of ϕ , the angle which the tangent to the curve makes with the horizon; for, in this case, $\bar{y}$ is given by the equation:—

$$\bar{y} \int_{y_0}^{y_1} \frac{dy}{ds} ds = \int_{y_0}^{y_1} y \frac{dy}{ds} ds, \quad \text{or} \quad \bar{y} \int_{y_0}^{y_1} dy = \int_{y_0}^{y_1} y dy,$$

7104. (By Professor BURNSIDE, M.A.)—Show that (1) the general differential equation of lines of the second order may be put under the form $\frac{d^3}{dx^3} \left\{ \left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} \right\} = 0$; (2) $\frac{1}{2} \frac{d^2}{dx^2} \left\{ \left(\frac{d^2y}{dx^2} \sin \omega \right)^{-\frac{2}{3}} \right\} = -K^{-\frac{2}{3}}$,

where K is that absolute invariant of the conic which depends upon the size only, the area being πK ; (3) hence also the general differential equation of all parabolas may be written

$$\frac{d^2}{dx^2} \left\{ \left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} \right\} = 0.$$

Solution by Prof. WOLSTENHOLME, M.A.; J. W. SHARPE, M.A.; and others.

If the equation of a line of the second order be

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0, \quad ax + hy + g + (hx + by + f) \frac{dy}{dx} = 0,$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{(hx + by + f) \left(a + h \frac{dy}{dx} \right) - (ax + hy + g) \left(h + b \frac{dy}{dx} \right)}{(hx + by + f)^2} \\ &= \frac{a(hx + by + f)^2 + b(ax + hy + g)^2 - 2h(ax + hy + g)(hx + by + f)}{(hx + by + f)^3} \\ &= \frac{(ab - h^2)(ax^2 + by^2 + c + 2fy + 2gx + 2hxy) - \Delta}{(hx + by + f)^3}, \end{aligned}$$

where Δ is the discriminant. Hence

$$\left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} = \frac{(hx + by + f)^2}{\Delta^{\frac{2}{3}}},$$

$$\begin{aligned} \frac{d}{dx} \left\{ \left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} \right\} &= 2\Delta^{-\frac{2}{3}} (hx + by + f) \left(h + b \frac{dy}{dx} \right) \\ &= 2\Delta^{-\frac{2}{3}} \{ h(hx + by + f) - b(ax + hy + g) \} = 2\Delta^{-\frac{2}{3}} \{ (h^2 - ab)x + hf - bg \}, \end{aligned}$$

whence

$$\frac{d^3}{dx^3} \left\{ \left(\frac{d^2y}{dx^2} \right)^{-\frac{2}{3}} \right\} = 0;$$

$$\text{and} \quad \frac{1}{2} \frac{d^2}{dx^2} \left\{ \left(\frac{d^2y}{dx^2} \sin \omega \right)^{-\frac{2}{3}} \right\} = \Delta^{-\frac{2}{3}} (h^2 - ab) (\sin \omega)^{-\frac{2}{3}} = -K^{-\frac{2}{3}}.$$

This seems to be the compactest form in which the general differential equation can be obtained. I have never seen it so arranged before, but cannot answer for its being a novelty. Surely there must be some compact way of expressing the other absolute invariant, depending upon the shape only, in terms of the differential coefficients, but I have not succeeded in discovering any at all worth recording. If ρ be the radius of curvature, s the arc, and a, b the semi-axes,

$$\frac{9 + \left(\frac{d\rho}{ds} \right)^2 - 3\rho \frac{d^2\rho}{ds^2}}{\rho^{\frac{5}{2}}} = 9(ab)^{-\frac{1}{2}} = 9K^{-\frac{1}{2}},$$

and is therefore

$$= -\frac{9}{2} \frac{d^2}{dx^2} \left\{ \left(\frac{d^2y}{dx^2} \sin \omega \right)^{-\frac{2}{3}} \right\};$$

$$\text{and} \quad \frac{1}{3} \left\{ 9 + \left(\frac{d\rho}{ds} \right)^2 \right\} + \frac{\left(\rho \frac{d^2\rho}{ds^2} \right)^2}{9 + \left(\frac{d\rho}{ds} \right)^2 - 3\rho \frac{d^2\rho}{ds^2}} = H^2,$$

where H is the absolute invariant $= \frac{2-e^2}{(1-e^2)^{1/2}}$, and e the excentricity.

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Let K_1^2, K_2^2, P^2 refer to the new axes. Then

$$K_1^2 = k_1^2 \cos^2 \theta + k_2^2 \sin^2 \theta - p^2 \sin 2\theta, \quad K_2^2 = k_1^2 \sin^2 \theta + k_2^2 \cos^2 \theta + p^2 \sin 2\theta,$$

$$P^2 = (k_1^2 - k_2^2) \sin \theta \cos \theta + p^2 \cos 2\theta,$$

Also (MINCHIN'S Statics) $a' = a \cos^2 \theta + b \sin^2 \theta + s \sin 2\theta$,

$$b' = a \sin^2 \theta + b \cos^2 \theta - s \sin 2\theta, \quad s' = (b-a) \sin \theta \cos \theta + s \cos 2\theta.$$

Hence, substituting and reducing, we have

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In any curve symmetrical about a vertical axis Y , the centre of gravity of the curve intercepted between by any two horizontal lines will bisect the line joining the middle points of the intercepts on these lines, if the density ρ be proportional to $\frac{dy}{ds}$, that is, the sine of ϕ , the angle which the tangent to the curve makes with the horizon; for, in this case, $\bar{y}$ is given by the equation:—

$$\bar{y} \int_{y_0}^{y_1} \frac{dy}{ds} ds = \int_{y_0}^{y_1} y \frac{dy}{ds} ds, \quad \text{or} \quad \bar{y} \int_{y_0}^{y_1} dy = \int_{y_0}^{y_1} y dy,$$

or
$$\bar{y} = \frac{1}{3} \left\{ \frac{y_1^2 - y_0^2}{y_1 - y_0} \right\} = \frac{y_1 + y_0}{2}.$$

In the case of the catenary,

$$\rho = K \frac{dy}{ds} = K \sin \phi = \frac{Ks}{(s^2 + c^2)^{1/2}},$$

s denoting the length of the arc from the lowest point of the catenary and c the parameter of the curve.

7135. (By D. EDWARDS.)—Three points are taken at random upon the surface of a given circle. Prove that (1) the probability that the triangle formed by joining them contains the centre of the given circle, is $\frac{1}{4}$; and (2) the probability that the circle drawn through the random points, which may, of course, cut the given circle, does not contain the centre of the given circle, is $\frac{8}{3\pi^2}$.

Solution by the PROPOSER; ELIZABETH BLACKWOOD; and others.

(1) The probability (p) in each case is the same, if one point be fixed on the circumference of the given circle. Let A, Fig. I., be the fixed point, B a random point, C the centre of given circle, $\angle ACB = \theta$, $CB = y$, $CA = a$. The shaded area will represent the number of favourable cases, and there-

fore
$$\pi a^2 p = 2 \int_0^\pi \int_0^a \frac{1}{2} a^2 \theta \cdot y \, d\theta \, dy, \text{ therefore } p = \frac{1}{4}.$$

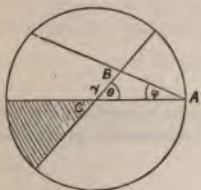


Fig. I.

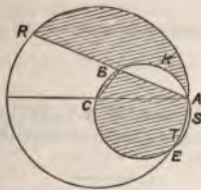


Fig. II. $y < a \cos \phi$.

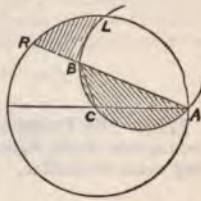


Fig. III. $y > a \cos \phi$.

(2) To avoid fractional expressions, refer B now to the point A. Let $AB = y$, $CA = a$, $\angle ACB = \theta$, $\angle CAB = \phi$, area $BCEA = u$, area $RBKA = v$, Fig. II.; and area $BCA = u'$, area $RBL = v'$, Fig. III.; and draw a circle through A, B, C. While $y < a \cos \phi$, the circle ABC cuts the fixed circle below the line CA, and above that line when $y > a \cos \phi$. If the second random point be situated in the shaded area, the circle through this point and B, A, does not contain C. Let R denote the radius of the circle ABC. Then $R = \frac{1}{2} a \operatorname{cosec} (\theta + \phi)$,

$$u = \pi R^2 - \text{seg. BKA} - \text{seg. ATES} = R^2 [\pi - \theta + \sin \theta \cos \theta - \pi \cos 2(\theta + \phi) + 2(\theta + \phi) \cos 2(\theta + \phi) - \sin 2(\theta + \phi)],$$

$$v = a^2 (\frac{1}{2} \pi - \phi - \sin \phi \cos \phi) - R^2 (\theta - \sin \theta \cos \theta);$$

whence, substituting functions of y and ϕ for functions of θ ,

$$u + v = a^2 (\pi - \theta - \phi - \sin \phi \cos \phi) + \frac{1}{2} \phi \operatorname{cosec}^2 \phi (a^2 \cos 2\phi - 2ay \cos \phi + y^2) \\ + \frac{1}{2} \operatorname{cosec} \phi (2ay - a^2 \cos \phi - y^2 \cos \phi).$$

Also $u' = R^2 (\pi - \theta + \sin \theta \cos \theta),$

$$v' = \int_{\phi}^{\pi - \theta - \phi} [2a^2 \cos^2 \psi - 2R^2 \sin^2 (\theta + \phi - \psi)] d\psi = (a^2 - R^2)(\pi - \theta - 2\phi) \\ - \frac{1}{2} a^2 \sin 2(\theta + \phi) - \frac{1}{2} R^2 \sin 4(\theta + \phi) - \frac{1}{2} a^2 \sin 2\phi + \frac{1}{2} R^2 \sin 2\theta;$$

hence

$$u' + v' = \frac{1}{2} \operatorname{cosec} \phi (2ay - y^2 \cos \phi - a^2 \cos \phi) + \frac{1}{2} \phi \operatorname{cosec}^2 \phi (a^2 - 2ay \cos \phi + y^2) \\ + a^2 (\pi - \theta - 2\phi) - a^2 \sin \phi \cos \phi.$$

The limits of ϕ are 0 and $\frac{1}{2}\pi$ and doubled. Then

$$\pi^2 a^4 p = 2 \int_0^{\frac{1}{2}\pi} \int_0^{\pi \cos \phi} (u + v) y d\phi dy + 2 \int_0^{\frac{1}{2}\pi} \int_{\pi \cos \phi}^{2a \cos \phi} (u' + v') y d\phi dy, \text{ or} \\ \pi^2 a^4 p = \int_0^{\frac{1}{2}\pi} \int_0^{2a \cos \phi} [\operatorname{cosec} \phi (2ay - y^2 \cos \phi - a^2 \cos \phi) + 2a^2 (\pi - \theta - \phi - \sin \phi \cos \phi) \\ + \phi \operatorname{cosec}^2 \phi (y^2 - 2ay \cos \phi) + a^2 \phi \operatorname{cosec}^2 \phi \cos 2\phi] y d\phi dy.$$

Now, $\int_0^{2a \cos \phi} \theta y dy = \int_0^{2a \cos \phi} \tan^{-1} \frac{y \sin \phi}{a - y \cos \phi} y dy \\ = a^2 (\pi - 2\phi) (1 + \frac{1}{2} \cos 2\phi) - a^2 \sin \phi \cos \phi,$

therefore $\pi^2 p = \int_0^{\frac{1}{2}\pi} [\frac{4}{3} \sin 2\phi - \frac{2}{3} \cot \phi + 4 \cos^2 \phi (\pi - \phi) + \frac{1}{3} \phi \operatorname{cosec}^2 \phi \\ + 2\phi + \frac{2}{3} \phi \cos 2\phi - 2\pi - \pi \cos 2\phi] d\phi \\ = \frac{2}{3} + \int_0^{\frac{1}{2}\pi} [\frac{4}{3} \sin 2\phi + \pi \cos 2\phi - \frac{4}{3} \phi \cos 2\phi] d\phi \\ = \frac{2}{3}, \text{ therefore } p = \frac{8}{3\pi^2}.$

7124. (By Professor GENÈSE, M.A.)—If three real triangles have the co-tangents of corresponding angles in arithmetical progression, prove that they must be similar.

Solution by E. W. SYMONS, M.A. ; Prof. NASH, M.A. ; and others.

Let $(l, m, n), (\lambda, \mu, \nu), (l', m', n')$ be the three sets of cotangents, then

$$mn + nl + lm = 1, \quad \&c. = 1, \quad \&c. = 1 \dots\dots\dots(1, 2, 3),$$

and $\lambda = \frac{1}{2} (l + l'), \quad \&c. = \&c.$

Substituting in (2), we get, by means of (1) and (3),

$$mn' + m'n + \dots = 2 = mn + m'n' + \dots \text{ by (1) and (3),}$$

therefore $(m - m')(n - n') + \dots = 0.$

Let $l - l' = \kappa_1 \dots\dots,$

then we have $\kappa_2 \kappa_3 + \dots = 0 \dots\dots\dots(4),$

and, by (3), $\kappa_1 (m + n) + \dots = 0, \text{ or } \kappa_1 \sin^2 A + \dots = 0 \dots\dots\dots(5).$

Substituting for κ_3 its value from (4), we get

$$\kappa_1^2 \sin^2 A + \kappa_1 \kappa_2 (\sin^2 A + \sin^2 B - \sin^2 C) + \kappa_2^2 \sin^2 B = 0,$$

or, if a, b, c be sides of the triangle opposite A, B, C ,

$\kappa_1^2 a^2 + 2\kappa_1 \kappa_2 ab \cos C + \kappa_2^2 b^2 = 0$, i.e., $(\kappa_1 a + \kappa_2 b \cos C)^2 + (\kappa_2 b \sin C)^2 = 0$; therefore, for real values, $\kappa_1 = 0$, $\kappa_2 = 0$, and by symmetry $\kappa_3 = 0$, therefore $l = l'$, &c., &c., and the triangles are all equiangular and similar.

[The PROPOSER remarks that (l, m, n) , &c. are the coordinates of points on the surface $yz + zx + xy = 1$, and the theorem follows immediately from the fact that this is *not* a ruled surface, that is, that three points on the surface cannot be in the same straight line.]

7148. (By Professor WOLSTENHOLME, M.A.)—Tangents are drawn to the curves $r^n \cos na = a^n \cos n(\theta + a)$ from the point $r = a$, $\theta = 0$: prove that the points of contact lie on the curve

$$\left(\frac{r}{a}\right)^{n+1} \frac{\sin n}{\sin \theta} - \left(\frac{r}{a}\right)^n \frac{\sin(n+1)\theta}{\sin \theta} + 1 = 0,$$

which involves an extraneous factor $r^2 - ar \cos \theta - a^2$.

Solution by Dr. CURTIS; Prof. MATZ, M.A.; and others.

In the class of curves defined by this equation, $-\frac{dr}{r d\theta} = \tan[n(\theta + a)]$;

consequently the angle between any radius vector and p the perpendicular from origin on the corresponding tangent $= n(\theta + a)$; it is therefore geometrically evident that, in the case of tangents from the point referred to,

$$r \cos[n(\theta + a)] = p = a \cos[(n+1)\theta + na],$$

$$\text{therefore } \frac{r}{a} = \frac{\cos[(n+1)\theta + na]}{\cos[n(\theta + a)]}, \text{ and } \frac{r^n}{a^n} = \frac{\cos[n(\theta + a)]}{\cos na};$$

$$\text{hence } \frac{r^{n+1}}{a^{n+1}} = \frac{\cos[(n+1)\theta + na]}{\cos na},$$

$$\text{therefore } \frac{r^n}{a^n} \sin[(n+1)\theta] - \frac{r^{n+1}}{a^{n+1}} \sin n\theta = \frac{\sin[(n+1)\theta] \cos[n(\theta + a)] - \cos[(n+1)\theta + na] \sin n\theta}{\cos na},$$

or, by expanding the terms containing a , $= \sin[(n+1)\theta - n\theta] = \sin \theta$; hence the result required. It is geometrically evident that r' , the distance of the extremity of the radius vector a , from any point on the curve, is given by the equation $r'^2 = a^2 + r^2 - 2ar \cos \theta$; and if, therefore, it is desired to exclude from the above locus the point from which the tangents are drawn, itself a point of contact of such a tangent, and corresponding to the condition $r' = 0$, it is plain that this factor must be suppressed.

[The PROPOSER calls this an extraneous factor, because all the other

points of contact lie on a single continuous curve not generally (if ever) passing through the point $(a, 0)$. Moreover, in his own investigation, he introduced this factor only to make the equation more compact. When $n = 2$, the locus is the straight line $2x + a = 0$; and, when $n = 3$, the hyperbola $3x^2 - y^2 + 2ax + a^2 = 0$.]

7107. (By Professor MALET, M.A.)—If the roots $x_1, x_2, x_3, x_4, x_5, x_6$ of the sextic equation $x^6 - p_1x^5 + p_2x^4 - p_3x^3 + p_4x^2 - p_5x + p_6 = 0$ be connected by the relation $x_1x_3x_5 = x_4x_5x_6$, prove that x_1, x_2, x_3 are the roots of the cubic

$$x^3 - \frac{1}{2} \left\{ p_1 + \left(p_1^2 - 4p_2 + \frac{4p_5}{\sqrt{p_6}} \right)^{\frac{1}{2}} \right\} x^2 + \frac{1}{2} \left\{ \frac{p_5}{\sqrt{p_6}} + \left(\frac{p_5^2}{p_6} - 4p_4 + 4p_1\sqrt{p_6} \right)^{\frac{1}{2}} \right\} x - \sqrt{p_6} = 0.$$

Solution by G. G. MORRICE, B.A.; SARAH MARKS; and others.

$$\begin{aligned} \text{Let } x_1 + x_2 + x_3 &= q, & x_1x_2 + x_2x_3 + x_3x_1 &= q_2, \\ x_4 + x_5 + x_6 &= r, & x_4x_5 + x_5x_6 + x_6x_4 &= r_2, \\ x_1^2x_3^2x_5^2 &= x_1x_2x_3x_4x_5x_6 = q_6. \end{aligned}$$

$$\begin{aligned} \text{We have } p_1 &= q_1 + r_1, & p_2 &= q_1r_1 + q_2 + r_2, & p_3 &= r_1q_2 + r_2q_1 + 2q_6, \\ p_4 &= \sqrt{q_6}(r_1 + q_1) + r_2q_2, & p_5 &= \sqrt{q_6}(r_2 + q_2). \end{aligned}$$

$$\text{Hence } p_2 = \frac{p_5}{\sqrt{q_6}} + (p_1 - q_1)q.$$

$$\text{Solving, } q_1 = \frac{1}{2} \left\{ p_1 \pm \left(p_1^2 - 4p_2 + \frac{4p_5}{\sqrt{p_6}} \right)^{\frac{1}{2}} \right\}.$$

$$\text{Similarly } q_2 = \frac{1}{2} \left\{ \frac{p_5}{\sqrt{p_6}} \pm \left(\frac{p_5^2}{p_6} - 4p_4 + 4p_1\sqrt{p_6} \right)^{\frac{1}{2}} \right\},$$

which gives for finding x_1, x_2, x_3 the equation given in the Question.

7158. (By W. J. C. SHARP, M.A.)—Find (1) the law of density, varying as the distance from the centre, that the centre of gravity of a hemisphere may be at a distance $\frac{r}{n}$ from the centre; and (2) the surface density that the centre of gravity may be the same.

Solution by Dr. CURTIS; MARY S. MEYER; and others.

1. Taking the centre of the sphere as origin and the axis of x perpendicular to the base of the hemisphere, and denoting by z the radius of any one of the homogeneous hemispherical shells of which the solid hemisphere is composed, and by ρ its density, we have $\rho = Kz^m$, and

$$\bar{X} \int_0^r 2\pi z^2 Kz^m dz = \int_0^r 2\pi z^2 Kz^m \frac{1}{2}z dz,$$

or $2\bar{X} \int_0^r z^{m+2} dz = \int_0^r z^{m+3} dz$, therefore $\bar{X} = \frac{(m+3)r}{2(m+4)}$;

but, by the condition of the question, we have

$$\bar{X} = \frac{r}{n}, \text{ therefore } \frac{n}{2} = \frac{m+4}{m+3}, \text{ hence } m = \frac{8-3n}{n-2}.$$

2. Taking origin and axis of X as before, and axis of Y in the base of the hemisphere, and denoting by θ the angle subtended at the centre of the sphere by the radius of any one of the homogeneous rings into which the hemispherical shell may be divided by a number of consecutive planes parallel to its base, the density will be represented by Kx^m , or by $Ka^m \cos^m \theta$, and the volume of the ring by $2\pi y r d\theta \times \tau$, τ being the uniform thickness of the shell, or by $2\pi r^2 \sin \theta d\theta \times \tau$, therefore

$$\bar{X} \int_0^{\frac{1}{2}\pi} 2\pi \tau Kx^m r^2 \sin \theta d\theta = \int_0^{\frac{1}{2}\pi} 2\pi \tau Kx^{m+1} r^2 \sin \theta d\theta,$$

or $\bar{X} \int_0^{\frac{1}{2}\pi} \cos^m \theta \sin \theta d\theta = r \int_0^{\frac{1}{2}\pi} \cos^{m+1} \theta \sin \theta d\theta$, or $\bar{X} = \frac{(m+1)r}{(m+2)}$;

but, by the condition of the question,

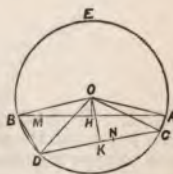
$$\bar{X} = \frac{r}{n}, \text{ therefore } n = \frac{m+2}{m+1} \text{ or } m = \frac{2-n}{n-1}.$$

6568. (By Professor SEITZ, M.A.)—A quadrilateral is formed by joining the ends of two chords, each of which is drawn at random through a random point within a circle; show that the average area of the quadrilateral is $\frac{17r^2}{4\pi}$.

Solution by the PROPOSER.

Let M and N be the two random points, AB and CD the random chords through them, O the centre of the circle, and $ABDC$ the quadrilateral. If the chords AB and CD intersect, the chords AC , AD , BC , BD will be the sides of the quadrilateral; and if C and D are both in the arc AEB , the chords AB , AD , CD , CB will be the sides. Draw OH , OK perpendicular to AB , CD .

Let $OA = r$, $\angle AOH = \theta$, $\angle COK = \phi$, $\angle KOH = \psi$, and ω = the angle OH makes with some fixed radius.



Then, from $\psi = 0$ to $\psi = \theta - \phi$, the area of the quadrilateral is

$$u_1 = \frac{1}{2}r^2 [\sin 2\phi - \sin 2\theta + 2 \cos \psi \sin (\theta - \phi)];$$

from $\psi = \theta - \phi$ to $\psi = \theta + \phi$, it is $u_2 = 2r^2 \sin \theta \sin \phi \sin \psi$; and from $\psi = \theta + \phi$ to $\psi = \pi$, it is $u_3 = \frac{1}{2}r^2 [\sin 2\phi + \sin 2\theta - 2 \cos \psi \sin (\theta + \phi)]$.

An element of the circle at M is $2r^2 \sin^2 \theta d\theta$, at N it is $2r^2 \sin^2 \phi d\phi$, and for elemental changes in the directions of AB and CD we have $d\omega$ and $d\psi$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ , 0 and θ , and doubled; of ω , 0 and 2π ; and of ψ , as stated above, and doubled. Hence, since the whole number of ways the two chords can be drawn is $\pi^2 r^4 \cdot \pi \cdot \pi = \pi^4 r^4$, the average area of the quadrilateral is

$$\begin{aligned} & \frac{4}{\pi^4 r^4} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \int_0^{2\pi} \left\{ \int_0^{\theta-\phi} u_1 d\psi + \int_{\theta-\phi}^{\theta+\phi} u_2 d\psi + \int_{\theta+\phi}^{\pi} u_3 d\psi \right\} \\ & \quad \times 2r^2 \sin^2 \theta d\theta \cdot 2r^2 \sin^2 \phi d\phi d\omega \\ &= \frac{32r^2}{\pi^3} \int_0^{\frac{1}{2}\pi} \int_0^{\theta} \left\{ (\pi - 2\theta) \sin \theta \cos \theta + (\pi - 2\phi) \sin \phi \cos \phi + 2 \sin^2 \theta + 2 \sin^2 \phi \right\} \\ & \quad \times \sin^2 \theta \sin^2 \phi d\theta d\phi \\ &= \frac{2r^2}{\pi^3} \int_0^{\frac{1}{2}\pi} \left\{ 8(\pi\theta - 2\theta^2) \sin \theta \cos \theta - 4(\pi - 2\theta)(2\sin^2 \theta - 3\sin^4 \theta) + 16\theta \right. \\ & \quad \left. + 16\theta \sin^2 \theta - 15 \sin \theta \cos \theta - 26 \sin^3 \theta \cos \theta \right\} \sin^2 \theta d\theta = \frac{17r^2}{4\pi}. \end{aligned}$$

7130. (By J. J. WALKER, M.A.)—Prove that the cotangent of the angle made by the right line through (x_1, y_1, z_1) , (x_2, y_2, z_2) with the side BC of the triangle of reference ABC is equal to

$$[(x_1 - x_2) \cos C - (y_1 - y_2) \cos B] + (x_1 - x_2) \sin A.$$

Solution by Rev. T. R. TERRY, M.A.; Prof. SCHEFFER; and others.

If P and Q be the two points, and θ the angle between the lines PQ and BC, we have at once

$$PQ \sin \theta = x_1 - x_2, \quad PQ \sin (B - \theta) = y_1 - y_2, \quad PQ \sin (C + \theta) = -(y_1 - y_2).$$

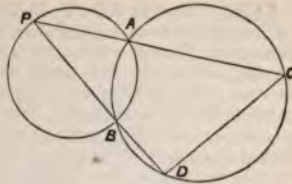
Therefore

$$\begin{aligned} (x_1 - x_2) \cos C - (y_1 - y_2) \cos B &= PQ \{ \cos C \sin (B - \theta) + \cos B \sin (C + \theta) \} \\ &= PQ \sin A \cos \theta = (x_1 - x_2) \sin A \cot \theta. \end{aligned}$$

7134. (By R. F. SCOTT, M.A.)—Two circles intersect at A and B. From any point P on one of them the lines PA, PB are drawn, and produced to meet the other circle again in C and D. Prove that, as the position of P varies, the straight line CD envelopes a circle concentric with the circle ABDC.

*Solution by R. KNOWLES, B.A., L.C.P. ;
E. W. SYMONS, M.A. ; and others.*

The angle APB is evidently constant for all positions of P, therefore the difference of the arcs AB and CD in circle ABCD is constant, and therefore CD is constant, and will consequently touch at its mid-point a circle concentric with ABDC.



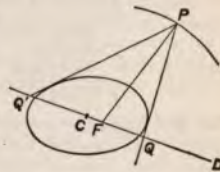
7136. (By T. WOODCOCK, B.A.)—Q is a point on the diameter parallel to the tangent at any point P of an ellipse. Prove that, if PQ always touches a given confocal ellipse, its length is constant.

Solution by E. W. SYMONS, M.A. ; CHRISTINE LADD, B.A. ; and others.

The analytical proof is easy, but, assuming Quest. 6962, we may argue thus:—PF the normal at P bisects angle QPQ' between two tangents from P to inner confocal, therefore

$$\frac{FP}{QP} = \frac{\mu}{CD},$$

μ being a constant, and CD conjugate to CP, therefore $\mu \cdot PQ = FP \cdot CD = \text{const.}$, therefore $PQ = \text{const.}$

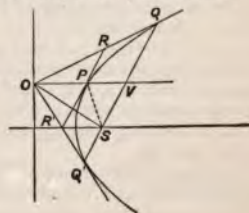


7153. (By R. F. SCOTT, M.A.)—A parabola always touches the rectangular axes so that its chord of contact is of constant length $2c$; prove that the locus of its focus is the curve $r = c \sin 2\theta$.

Solution by J. P. JOHNSTON, B.A. ; J. S. JENKINS, M.A. ; and others.

The chord of contact QQ' passes through the focus S, and is perpendicular to the line joining S to the origin O. Draw the diameter OPV through O,

$\angle \theta = (\angle SOQ') = \angle SQO = \angle QOV$,
therefore $\angle SVO = 2\theta$,
 $OS = OV \sin 2\theta$
 $= c \sin 2\theta$.



6412. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—The substitution 56781234 has twelve square roots, each having two circular factors of four elements. The powers and products of these twelve θ , ϕ , &c., viz., θ^2 , $\theta\phi$, $\phi\theta$, &c., are a transitive group G of 32 substitutions, of which the title is $8 \cdot 4 = 1 + 12_{44} + 13_{222} + 6_{221111}$; $Q = 105$: i.e., the group contains besides unity 12345678 and the 12, θ , ϕ , &c., 13 square roots of unity having no element undisturbed, and six square roots of unity having four undisturbed elements. The group of 32 has, including G, 105 equivalents, G, $\alpha G\alpha^{-1}$, $\beta G\beta^{-1}$, &c. Construct a transitive group H of the same order 32, having the same title, except that the number of its equivalents is $Q = 630$, in which the substitutions 12_{44} are not all square roots of the same substitution of the second order. [The interest of the question lies in the fact that this is the simplest of all cases of two transitive groups whose titles differ in nothing except the number Q.]

Solution by the PROPOSER.

The former of these groups is the following:—

12345678	12347856	14325876	14327658
A 23416785	C 23418567	43218765	43216587
α 34127856	34125678	32147658	32145876
B 41238567	D 41236785	21436587	21438765
56781234	G 78561234	58761432	I 76581432
E 67852341	85672341	87654321	J 65874321
78563412	H 56783412	76583214	K 58763214
F 86674123	67854123	65872143	L 87652143

The first half is an intransitive group of 16. The twelve substitutions marked with capitals are all square roots of α . The number of equivalents is 105, that of the substitutions of the form of α .

The second group is the following:—

12345678	12346587	12347856	12348765
α 21436587	21435678	ν 21438765	ρ 21437856
β 34127856	λ 34128765	34125678	ϕ 34126587
γ 43218765	μ 43217856	π 43216587	43215678
56781234	A 65871234	E 78561234	I 87651234
65872143	B 56782143	F 87652143	J 78562143
78563412	C 87653412	G 56783412	K 65873412
87654321	D 78564321	H 65874321	L 56784321

The first half is a group $1\alpha\beta\gamma$ followed by three derivatives completing an intransitive group of 16. ABCD are four square roots of α , EFGH of β , and IJKL of γ .

The intransitive group of 16 can be written commencing with the group $1\alpha\lambda\mu$. There must therefore be another derivate completing a transitive group of 32, which has four square roots of each of α , λ , μ .

The same intransitive of 16 can be written commencing with the group $1\beta\pi\nu$, or with $1\phi\rho$, or with $1\lambda\rho\pi$, or with $1\nu\mu\phi$; and with each of these arrangements is given a different derivate completing a transitive group of 32, containing four square roots of each substitution of the opening triplet $\beta\pi\nu$, &c.

The intransitive group of 16 has 105 equivalents. There are consequently $6 \cdot 105 = 630$ equivalent groups of 32.

The title of the former group is given in the abstract of my Treatise,

"The Complete Theory of Groups," at p. 144 of the *Proceedings of the Manchester Lit. Phil. Soc.*, 1863. It is no subject of wonder that the treatise was not printed, after the refusal of the Royal Society to print my much more original, if not more important theory of the Polyedra; but it is perhaps to be regretted that the work on Groups has disappeared from the archives of the Philosophical Society of Manchester, and cannot, after much enquiry, be found. I have no copy of my MS.

The title of the second group is given in a table of *Addenda*, at p. 172 of the same *Proceedings*, for 1865; but with the error $Q = 210$, instead of $Q = 630$. My oversight was the confinement of my attention to the two groups above noted $1a\beta\gamma$ and $1a\lambda\mu$.

All the six equivalent groups completed on the intransitive of 16 are obtained by the six multipliers 87654321, 87663421, 86672341, 85762431, 86754231, 86573241.

It was a surprise to me to find that two non-equivalent transitive groups may have titles identical in everything but the number Q .

6149. (By S. TEBAY, B.A.)—Required a direct general solution of the equation $N^2 = H + 3st(2m-s-t)$, without assuming particular values of s, t to make the absolute term a square. [This problem solves the celebrated diophantine problem,—“To find n numbers such that, if each be taken from the cube of their sum, the remainders shall be cubes.” Mr. TEBAY’s results are

$$24r^2(3mr^2+1)s = 9Hr^6-4, \quad (3r^2s+1)t = 2m-s.]$$

Solution by the PROPOSER; W. NICHOLLS, B.A.; and others.

Let $3st(2m-s-t) = 9r^2s^2t^2$; then $t = \frac{2m-s}{3r^2s+1}$. Substituting this value of t , we have

$$N^2(3r^2s+1)^2 = 9r^2s^4 - 36mr^2s^3 + 9r^2(Hr^2 + 4m^2)s^2 + 6Hr^2s + H = 9r^2(s^2 - 2ms + \frac{1}{2}Hr^2)^2.$$

Therefore $24r^2(3mr^2+1)s = 9Hr^6-4$.

This solves the problem. Its application to the diophantine problem in question is as follows.

Let a be the sum of n numbers, x, y, z , &c., such that a^3-x, a^3-y, a^3-z , &c. shall be cubes; then, by addition, $na^3-a = a^3(n-a^{-2})$ is the sum of n cubes, or $n-a^{-2} = n$ cubes.

Let $m-s, m-t, s+t-m$ be the roots of three of them; then, putting $H = n-(m^2 + \text{remaining cubes})$, we have $a^{-2} = H + 3st(2m-s-t)$, which is the proposed form.

This is a remarkable application of the diophantine analysis; but it is only so far general that $3st(2m-s-t)$ is always a square. The following particular solutions are deserving of notice. For s, t write $\lambda+s, \mu+t$, then $a^{-2} = H + 3\lambda\mu(2m-\lambda-\mu) + 3\mu(2m-2\lambda-\mu-s)s$

$$+ 3(\lambda+s)(2m-\lambda-2\mu-s-t)t.$$

This equation can be solved whenever $H + 3\lambda\mu(2m-\lambda-\mu)$ can be made a square.

Let $m = 2$, $n = 3$, $\lambda = 1$, $\mu = 1$; then
 $a^{-2} = 1 + 3(1-s)s + 3(1+s)(1-s-t)t$.

Let $P^2 = 1 + 3(1-s)s = (1+ps)^2$;
 $s = \frac{3-2p}{p^2+3}$, $P = \frac{3+3p-p^2}{p^2+3}$;
 then $a^{-2} = P^2 + 3(1+s)(1-s-t)t = (P+qt)^2$;

Therefore $a^{-2} = \frac{3(1-s^2)-2Pq}{q^2+3(1+s)}$, $a(P+qt) = 1$.
 therefore $t = \frac{3(1-s^2)-2Pq}{q^2+3(1+s)}$, $a(P+qt) = 1$.

Let $p = q = 1$; then $s = \frac{1}{2}$, $P = \frac{2}{3}$, $t = \frac{5}{78}$, $a = \frac{1}{2} \frac{2}{3} = \cdot 76$, $x = \cdot 081065$,

$y = \cdot 253783$, $z = \cdot 425152$.

Again, let $m = 1$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, then
 $a^{-2} = \frac{2}{3} + \frac{1}{12}(7-6s)s + \frac{1}{2}(1+3s)(4-3s-3t)t$.

As before, we find

$$s = \frac{7-36p}{6(2p^2+1)}, \quad P = \frac{9+7p-18p^2}{6(2p^2+1)}, \quad t = \frac{\frac{1}{2}(1+3s)(4-3s)-2Pq}{q^2+1+3s},$$

$$a(P+qt) = 1.$$

Let $p = \frac{1}{10}$, $q = 0$, then $s = \frac{5}{9}$, $t = \frac{7}{9}$, $a = \frac{2}{14}$; and the three roots
 are $\frac{2}{13}$, $\frac{2}{13}$, $\frac{1}{13}$, which are Dr. HART's results.

6784. (By A. McINTOSH, B.A.)—Prove that (1) from any point of a Folium of Descartes which is not situated on the loop, two real tangents can be drawn to the curve; (2) if P be the point, PT, PT' the tangents; (3) N the node, NT and NT' make equal angles with the nodal tangents; (4) if PN be produced on to meet the chord TT', it is bisected in N, and it also bisects TT'; (5) if the tangent (PQ) to the curve at the point P be produced on to meet the chord TT' in Q, PQ is bisected by the curve; (6) the chord TT' makes with the nodal tangents a triangle of constant area, and therefore touches a rectangular hyperbola having those tangents for asymptotes; (7) if the chord TT' meet the loop in a second point R, PR being the line joining the primary point to this, PR also touches the rectangular hyperbola in (5); (8) the half of the chord TT' remote from the loop subtends the same angle as the segment RT' within the loop at the node; (9) the middle point of TT' lies on an identical folium having its loop in the opposite quadrant; (10) if from the middle point of TT' two tangents be drawn to the Folium in (9), the line joining the points of contact will pass through P; (11) the point of trisection of any tangent chord such as PT, most remote from the point of contact T, lies on a Folium of Descartes; (12) the centroid of the triangle PTT' is such a point; (13) in nodal cubics in general, NT and NT' form a harmonic pencil with the nodal tangents; (14) also TT' and PR touch a conic section which touches the nodal tangents.

Solution by G. F. WALKER, M.A.; Prof. NASH, M.A.; and others.
 The equation to the curve being $x^3 + y^3 = 3axy$, let (α, β) be the poi

P; then the tangent at P is $(a^2 - a\beta)x + (\beta^2 - a\alpha)y = a\alpha\beta$, and the conic $(x^2 - ay)\alpha + (y^2 - ax)\beta = a\alpha\beta$ will pass through the points of contact of the tangents. The lines joining these points to N will be

$$3axy [ax^2 + \beta y^2 - a\alpha y] = a(\alpha y + \beta x)(x^3 + y^3),$$

and, remembering that $a\beta$ is on the curve, this becomes, by eliminating a ,

$$(\alpha y - \beta x)^2 (ax^2 + \beta y^2) = 0.$$

The equation to NT and NT' is therefore $ax^2 + \beta y^2 = 0$, which proves (2), and they will be real if α and β have different signs, which proves (1).

The lines NT, NT' will meet the curve at points lying on the conic $\alpha y + \alpha y + \beta x = 0$, and if $lx + my = 1$ be the equation to TT',

$$xy + (\alpha y + \beta x)(lx + my) = 0$$

must be identical with $ax^2 + \beta y^2$; therefore

$$1 + \beta m + \alpha l = 0, \text{ and } \frac{\beta l}{\alpha} = \frac{\alpha m}{\beta} = \kappa \text{ suppose.}$$

Then $\kappa \left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} \right) + 1 = 0$, therefore $\kappa = -\frac{1}{3a}$,

since $a\beta$ is on the curve, and the equation to TT' is $x \frac{\alpha}{\beta} + y \frac{\beta}{\alpha} = -3a$, and this touches $4xy = 9a^2$, which proves (5).

This hyperbola passes through the point $x = y = \frac{3}{2}a$, the vertex of the loop, and has the same tangent $x + y = 3a$, which proves (6).

The equation to PN is $\alpha y - \beta x$, and where this meets TT' we have

$$x \left[\frac{\alpha}{\beta} + \frac{\beta^2}{\alpha^2} \right] = -3a, \text{ or } x = -\alpha,$$

which proves the first part of (3).

The conditions of the middle point of TT' may be found thus: the x 's of TT' are found by eliminating y between

$$xy + \alpha y + \beta x = 0 \text{ and } ax^2 + \beta y^2 = 0,$$

and are therefore given by the equations

$$ax^2 + \frac{\beta^3 x^2}{(x + \alpha)^2} = 0, \quad \alpha(x + \alpha)^2 + \beta^3 = 0, \quad x^2 + 2ax + 3a\beta = 0.$$

The coordinate of the middle point is therefore $-\alpha$, which proves the rest of (3) and (9).

This also proves (10), as the two folia are symmetrical with respect to the lines $x \pm y$, and the chord will not only pass through P, but be bisected at it and be parallel to TT'.

The coordinates of Q are found to be $-\alpha \frac{\alpha^3 - 3\beta^3}{\alpha^3 - \beta^3}$, $-\beta \frac{3\alpha^3 - \beta^3}{\alpha^3 - \beta^3}$, remembering that $a\beta$ is on the curve, and therefore the coordinates of the middle point of PQ are $\frac{\alpha\beta^3}{\alpha^3 - \beta^3}$, $-\frac{\alpha^3\beta}{\alpha^3 - \beta^3}$, which is a point on the folium; which proves (4). The equation to the lines NT, NT', NR is

$$x^3 + y^3 + xy \left[x \frac{\alpha}{\beta} + y \frac{\beta}{\alpha} \right] = 0, \text{ or } (ax^2 + \beta y^2)(\beta x + \alpha y) = 0.$$

The equation to NR is therefore $\beta x + \alpha y$, which proves (8), since the equation to the line joining N to the middle point of TT' is $\beta x - \alpha y$, and these make equal angles with $x = 0$, as also do NT and NT'.

The equation to RP is of the form

$$x \frac{\alpha}{\beta} + y \frac{\beta}{\alpha} + 3a + \lambda (\beta x + \alpha y) = 0,$$

and passes through (α, β) .

Hence $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} + 3a + 2\lambda\alpha\beta = 0$, therefore $\lambda = -\frac{3a}{\alpha\beta}$,

since (α, β) is on the curve; therefore PR is

$$x \frac{\alpha}{\beta} + y \frac{\beta}{\alpha} + 3a - 3a \left(\frac{x}{\alpha} + \frac{y}{\beta} \right) = 0, \quad x \frac{\alpha^2 - 3a\beta}{\alpha\beta} + y \frac{\beta^2 - 3a\alpha}{\alpha\beta} + 3a = 0,$$

or, since (α, β) is on the curve, this becomes

$$x \frac{\beta^2}{\alpha^2} + y \frac{\alpha^2}{\beta^2} = 3a,$$

which touches the hyperbola $4xy = 9a^2$, which proves (7).

The coordinates of T are found, from $x(\alpha)^{\frac{1}{2}} = -y(-\beta)^{\frac{1}{2}}$ and the curve to be $-\frac{\alpha^{\frac{3}{2}} + (-\beta)^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}}$, $\frac{\alpha^{\frac{3}{2}} + (-\beta)^{\frac{3}{2}}}{(-\beta)^{\frac{1}{2}}}$, remembering that $\alpha\beta$ is on the curve

hence the coordinates of the required point of trisection are

$$\frac{\alpha^{\frac{3}{2}} - (-\beta)^{\frac{3}{2}}}{3\alpha^{\frac{1}{2}}}, \quad \frac{\alpha^{\frac{3}{2}} - (-\beta)^{\frac{3}{2}}}{3(-\beta)^{\frac{1}{2}}},$$

which is a point on the folium $x^3 + y^3 + axy = 0$, which proves (11).

The centroid of TPT' evidently lies on this folium, since it is found by producing PN to $\frac{1}{3}$ of its length, [this proves (12)], and on the sinuous branch. The results (13) and (14) may be found by projection from (2), (5), and (7).

6858. (By Professor WOLSTENHOLME, M.A.)—The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is cut at right angles by a concentric (not confocal) conic; prove that (1) the common chords which are not diameters touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2 - b^2};$$

(2) the locus of the intersection of normals to the hyperbola at the ends of such a common chord is the curve

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2 \frac{x^2 + y^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 - b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2,$$

whose asymptotes are $\frac{x}{a} \pm \frac{y}{b} \pm \left(\frac{a^2 + b^2}{a^2 - b^2} \right)^{\frac{1}{2}} = 0$.

Solution by the PROPOSER; G. HEPPLE, M.A.; and others.

The equation of a conic cutting $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at right angles will be

$$\frac{x^2}{a^2} + \frac{2xy}{ab} \lambda + \frac{y^2}{b^2} = \frac{a^2 + b^2}{a^2 - b^2};$$

or, putting $\lambda = \sin 2\alpha$, we have, at their common points,

$$\frac{x^2}{a^2} + \frac{2xy}{ab} \sin 2\alpha + \frac{y^2}{b^2} + \cos 2\alpha \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = \frac{a^2 + b^2}{a^2 - b^2} + \cos 2\alpha,$$

which is $\left(\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha \right)^2 = \frac{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}{a^2 - b^2},$

or is the equation of two common chords not diameters.

The envelope of these chords is

$$\left(\frac{x^2}{a^2} - \frac{a^2}{a^2 - b^2} \right) \left(\frac{y^2}{b^2} - \frac{b^2}{a^2 - b^2} \right) = \frac{x^2 y^2}{a^2 b^2}, \text{ or } \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 - b^2}.$$

The normals to the hyperbola are tangents to the orthogonal conic; hence their intersection is the pole of

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = \left(\frac{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}{a^2 - b^2} \right)^{\frac{1}{2}}$$

with respect to the conic $\frac{x^2}{a^2} + \frac{2xy}{ab} \sin 2\alpha + \frac{y^2}{b^2} = \frac{a^2 + b^2}{a^2 - b^2}.$

The equations for the point are, therefore,

$$\frac{\frac{x}{a} + \frac{y}{b} \sin 2\alpha}{\cos \alpha} = \frac{\frac{x}{a} \sin 2\alpha + \frac{y}{b}}{\sin \alpha} = \frac{a^2 + b^2}{\left\{ (a^2 - b^2) (a^2 \cos^2 \alpha + b^2 \sin^2 \alpha) \right\}^{\frac{1}{2}}},$$

whence $\frac{x}{a \cos \alpha} = \frac{-y}{b \sin \alpha} = \cos 2\alpha \left\{ (a^2 - b^2) (a^2 \cos^2 \alpha + b^2 \sin^2 \alpha) \right\}^{\frac{1}{2}},$

and the locus is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^{\frac{1}{2}} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{\frac{a^2 + b^2}{a^2 - b^2} \left\{ (a^2 - b^2) \right\}^{\frac{1}{2}}} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^{\frac{1}{2}},$$

or $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2 \frac{x^2 + y^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 - b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2;$

and for the asymptotes we have

$$\left\{ \frac{\left(\frac{x}{a} - \frac{y}{b} \right)^2}{\frac{a^2 + b^2}{a^2 - b^2}} \frac{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2}{\left(\frac{x}{a} + \frac{y}{b} \right)^2 (x^2 + y^2)} \right\} = \frac{a^2 + b^2}{a^2 - b^2} \dots (1);$$

and, similarly, $\left(\frac{x}{a} + \frac{y}{b} \right)^2 = \frac{a^2 + b^2}{a^2 - b^2} \dots \dots \dots (2)$

The curve consists of an acnode (quadruple point) at the origin and four hyperbolic branches.

7086. (By G. F. WALKER, M.A.)—If the surface

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2Ax + 2By + 2Cz + D = 0$$

represent a paraboloid of revolution, prove that, provided f, g, h be finite,

its focus and directrix plane are given by the equations

$$\begin{aligned} f \left[x + \frac{2A}{a+b+c} \right] &= g \left[y + \frac{2B}{a+b+c} \right] \\ &= h \left[z + \frac{2C}{a+b+c} \right] = fg h \frac{2 \frac{A^2+B^2+C^2}{a+b+c} - D}{2 (Ag h + Bh f + Cf g)}, \\ 2 (Ag h + Bh f + Cf g) (g h x + h f y + f g z) \\ &+ (A^2 + B^2 + C^2) f g h + D (g^2 h^2 + h^2 f^2 + f^2 g^2) = 0. \end{aligned}$$

Solution by T. WOODCOCK, B.A. ; D. EDWARDS ; and others.

Putting $ax^2 + \dots + 2fyz + \dots + 2Ax + \dots + D$

$$\equiv (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - (lx + my + nz - p)^2,$$

where $l^2 + m^2 + n^2 = 1$, $(\alpha\beta\gamma)$ being the focus, and $lx + my + nz = p$ the directrix plane, we get

$$\frac{a}{1-l^2} = \dots = \dots = \frac{-f}{mn} = \dots = \dots = \frac{A}{pl-\alpha} = \dots = \dots = \frac{D}{\alpha^2 + \beta^2 + \gamma^2 - p^2} = \lambda mn.$$

Therefore

$$a + b + c = 2\lambda,$$

$$A^2 + B^2 + C^2 = \lambda^2 \{ p^2 + \alpha^2 + \beta^2 + \gamma^2 - 2p(l\alpha + m\beta + n\gamma) \},$$

$$Ag h + Bh f + Cf g = \lambda^3 lmn \{ p - (l\alpha + m\beta + n\gamma) \},$$

and

$$g^2 h^2 + h^2 f^2 + f^2 g^2 = \lambda^4 l^2 m^2 n^2;$$

$$\text{therefore } f \left(a + \frac{A}{\lambda} \right) = g \left(b + \frac{B}{\lambda} \right) = h \left(c + \frac{C}{\lambda} \right) = -p\lambda lmn$$

$$\begin{aligned} &= \frac{2 \frac{A^2 + B^2 + C^2}{a+b+c} - D}{2 (Ag h + Bh f + Cf g)} f g h, \end{aligned}$$

and the equation $lx + my + nz - p = 0$ may be written in the required form by multiplying the left side by

$$2\lambda^3 l^2 m^2 n^2 \{ p - (l\alpha + m\beta + n\gamma) \}.$$

5622. (By J. L. MCKENZIE, B.A.)—If s_r denote the sum of the r^{th} powers of the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$; find the value of the determinant

$$\begin{vmatrix} s_0 & s_1 & \dots & s_{i-1} & s_{i+1} & \dots & s_n \\ s_1 & s_2 & \dots & s_i & s_{i+2} & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s_{k-1} & s_k & \dots & s_{k+i-2} & s_{k+i} & \dots & s_{n+k} \\ s_{k+1} & s_{k+2} & \dots & s_{k+i} & s_{k+i+2} & \dots & s_{n+k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n+1} & \dots & s_{n+i-1} & s_{n+i+1} & \dots & s_{2n} \end{vmatrix}$$

Solution by R. F. SCOTT, M.A.; W. J. C. SHARP, M.A.; and others.

If there be n quantities $a_1, a_2, \dots, a_n$, then $\Pi (a_p - a_q)$ is to mean the product of all the differences of the quantities $a_1, \dots, a_n$. It is supposed that p is always less than q . This being so, we have

$$\begin{vmatrix} a_1^n & a_2^n & \dots & a_n^n & x^n & 0 \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} & x^{n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n & x & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{vmatrix} = (-1)^n \Pi (a_p - a_q) (x - a_1)(x - a_2) \dots (x - a_n);$$

$$\begin{vmatrix} a_1^n & a_2^n & \dots & a_n^n & 0 & y^n \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} & 0 & y^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n & 0 & y \\ 1 & 1 & \dots & 1 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{vmatrix} = (-1)^{n-1} \Pi (a_p - a_q) (y - a_1)(y - a_2) \dots (y - a_n);$$

Multiply these two determinants by rows, $s_i = a_i^n + a_i^{n-1}x + \dots + a_i$

$$\begin{vmatrix} s_{2n} & s_{2n-1} & \dots & s_n & x^n \\ s_{2n-1} & s_{2n-2} & \dots & s_{n-1} & x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n-1} & \dots & s_0 & 1 \\ y^n & y^{n-1} & \dots & 1 & 0 \end{vmatrix} = -\Pi (a_p - a_q)^2 \cdot (x - a_1)(x - a_2) \dots (x - a_n) \times (y - a_1)(y - a_2) \dots (y - a_n);$$

Now let $f(z) = (z - a_1)(z - a_2) \dots (z - a_n) = z^n + p_1 z^{n-1} + \dots + p_n$.

Equate the coefficients of $x^k y^k$ on both sides, then, with the exception of an interchange of rows and columns,

$$\text{required determinant} = (-1)^{n(n-1)/2} \Pi (a_p - a_q)^2 \cdot p_{n-1} p_{n-2} \dots$$

7190. (By Professor WOLSTENHOLME, M.A.)—If x, y, z be three quantities satisfying the two symmetrical equations

$$yz + zx + xy = 0, \quad x^2 + y^2 + z^2 + 4xyz = 0;$$

prove that (1) they will also satisfy one of the two pairs of semi-symmetrical expressions

$$y^2z + z^2x + x^2y = (y - z)(z - x)(x - y), \quad = +xyz,$$

$$yz^2 + zx^2 + xy^2 = (y - z)(z - x)(x - y), \quad = -xyz;$$

and (2) one set of the following equations will also be satisfied:—

$$(x^2 + yz - y^2 = 0, \quad y^2 + zx - z^2 = 0, \quad z^2 + xy - x^2 = 0);$$

$$(x^2 + yz - z^2 = 0, \quad z^2 + zx - x^2 = 0, \quad x^2 + xy - y^2 = 0).$$

Solution by Professor CAYLEY, F.R.S.

The two symmetrical equations represent a conic and a cubic respectively; they intersect therefore in 6 points, and if we denote by α a root of the equation $u^3 + u^2 - 2u - 1 = 0$, then the other two roots of this equation are $\beta, \gamma = -1 - \frac{1}{\alpha}, \gamma = \frac{-1}{\alpha+1}$; viz., if $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$, then we have

$$(u - \alpha) \left(u + 1 + \frac{1}{\alpha} \right) \left(u + \frac{1}{\alpha+1} \right) = u^3 + u^2 - 2u - 1,$$

an identity which is easily verified. It may be remarked that, if $\phi\alpha = -1 - \frac{1}{\alpha}$, then $\phi^2\alpha = \frac{-1}{\alpha+1}$, $\phi^3\alpha = \alpha$; the left-hand side of the last mentioned equation thus is $(u - \alpha)(u - \phi\alpha)(u - \phi^2\alpha)$, which remains unaltered when α is changed into $\phi\alpha$ or $\phi^2\alpha$. Then the coordinates of the six points of intersection can be expressed indifferently in terms of any one of the roots (α, β, γ), viz., the coordinates are

$$(\alpha^2 - 1, -\alpha, -1), (-1, \alpha^2 - 1, -\alpha), (-\alpha, -1, \alpha^2 - 1) \dots (1, 2, 3),$$

$$(\alpha^2 - 1, -1, -\alpha), (-\alpha, \alpha^2 - 1, -1), (-1, -\alpha, \alpha^2 - 1) \dots (4, 5, 6);$$

or they are equal to the like expressions in β and in γ respectively; say these are the coordinates of the points 1, 2, 3, 4, 5, 6 respectively, as shown by the attached numbers. Thus, writing $x, y, z = \alpha^2 - 1, -\alpha, -1$, we find $yz + zx + xy = \alpha - \alpha^2 + 1 - \alpha^3 + \alpha = -(\alpha^3 + \alpha^2 - 2\alpha - 1) = 0$,

$$x^3 + y^3 + z^3 + 4xyz = (\alpha^2 - 1)^3 - \alpha^3 - 1 + 4\alpha(\alpha^2 - 1),$$

$$= \alpha^6 - 3\alpha^4 + 3\alpha^2 + 3\alpha^2 - 4\alpha - 2 = (\alpha^3 + \alpha^2 - 2\alpha - 1)(\alpha^3 - \alpha^2 + 2) = 0,$$

which verifies the formulæ for the six points of intersection. Take, again, $x, y, z = \alpha^2 - 1, -\alpha, -1$; then we find

$$y^2z + zx^2 + xy^2 = -\alpha - (\alpha^2 - 1)^2 + \alpha^2(\alpha^2 - 1) = \alpha^2 - \alpha - 1,$$

$$y^2z + z^2x + x^2y = -\alpha^2 + (\alpha^2 - 1) - \alpha(\alpha^2 - 1)^2 = -\alpha^5 + 2\alpha^3 - \alpha - 1.$$

Or, since $\alpha^3 = -\alpha^2 + 2\alpha + 1$, and thence $\alpha^4 = 3\alpha^2 - \alpha - 1$, $\alpha^5 = -4\alpha^2 + 5\alpha + 3$, the last equation becomes $y^2z + z^2x + x^2y = 2\alpha^2 - 2\alpha - 2$. We have also

$$xyz = \alpha^3 - \alpha = -\alpha^2 + \alpha + 1;$$

hence the point in question is situate on each of the cubics

$$yz^2 + zx^2 + xy^2 + xyz = 0, \quad y^2z + z^2x + x^2y + 2xyz = 0,$$

$$y^2z + z^2x + x^2y - 2(yz^2 + zx^2 + xy^2) = 0;$$

and this, of course, shows the points 1, 2, 3 are all three of them situate upon each of the three cubics; and in precisely the same manner it appears that the points 4, 5, 6 are all three of them situate on each of the three cubics $yz^2 + zx^2 + xy^2 + 2xyz = 0$, $y^2z + z^2x + x^2y + xyz = 0$,

$$y^2z + z^2x + xy^2 - 2(yz^2 + zx^2 + x^2y) = 0.$$

Again, from the values $x, y, z = \alpha^2 - 1, -\alpha, -1$, we have

$$x^2 + yz - y^2 = 0, \quad y^2 + zx - z^2 = 0, \quad z^2 + xy - x^2 = 0;$$

viz., the point 1 lies on each of these conics; and similarly the point 2 lies on each of the same conics; and the point (3) lies on each of the same conics; that is, the conics in question have in common the points 1, 2, 3.

In like manner, the conics

$$x^2 + yz - z^2 = 0, \quad y^2 + zx - x^2 = 0, \quad z^2 + xy - y^2 = 0,$$

have in common the points 4, 5, 6.

The general result is that the given conic and cubic meet in six points forming two groups of points (1, 2, 3) and (4, 5, 6); through the points (1, 2, 3) we have three cubics and three conics; and through the points (4, 5, 6) we have three cubics and three conics.

[If in the equation $x^3 + x^2 - 2x - 1 = 0$, whose roots are $\alpha, \phi(\alpha), \phi^2(\alpha)$, we put $x = 2 \cos \theta$, the equation becomes

$$2(3 \cos \theta + \cos 3\theta) + 2(1 + \cos 2\theta) - 4 \cos \theta - 2 = 0,$$

or $2 \cos 3\theta + 2 \cos 2\theta + 2 \cos \theta = 0$, or $\frac{\sin \frac{7}{2}\theta}{\sin \frac{1}{2}\theta} = 0$;

or the three roots are $2 \cos \frac{2}{3}\pi, 2 \cos \frac{4}{3}\pi, 2 \cos \frac{6}{3}\pi$. The two equations $yz + zx + xy = 0, x^3 + y^3 + z^3 + 3xyz = 0$ are satisfied if $x : y : z =$ these three roots in any order, giving the six solutions. The semi-symmetrical systems are satisfied, the one by

$$x : y : z, \text{ or } y : z : x, \text{ or } z : x : y = \cos \frac{2}{3}\pi : \cos \frac{4}{3}\pi : \cos \frac{6}{3}\pi;$$

and the other by

$$z : y : x \text{ or } y : x : z, \text{ or } x : z : y = \cos \frac{2}{3}\pi : \cos \frac{4}{3}\pi : \cos \frac{6}{3}\pi.]$$

7169. (By Professor HUDSON, M.A.)—The sum of the three sides of a right-angled spherical triangle is a quadrant, prove that the least value of the hypotenuse is $\cos^{-1} \frac{1}{2}$, and that in this case the spherical excess is $\sin^{-1} \frac{1}{2}$.

Solution by W. H. BLYTHE, M.A.; H. K. MOORE, B.A.; and others.

Since the triangle is right-angled, and $a + b + c = \frac{1}{2}\pi$, we have

$$\cos c = \cos a \cos b = \cos a \sin(a + c) = \cos a \sin a \cos c + \cos^2 a \sin c;$$

hence $\tan c = \sec^2 a - \tan a = (\tan a - \frac{1}{2})^2 + \frac{3}{4}$;

therefore $\tan c$ is evidently a minimum when $\tan a = \frac{1}{2}$, which gives $\tan c = \frac{3}{4}$, or $c = \cos^{-1} \frac{3}{5}$.

In this particular case, also, we find

$$b = \frac{1}{2}\pi - \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{3}{4} \text{ or } b = \tan^{-1} \frac{1}{2} = a.$$

We find thus that $B = A$, and the sine of the spherical excess, or $\sin(A + B + C - \pi)$ reduces to $-\cos 2A$, C being a right angle; thus we obtain

$$1 - 2 \cos^2 A = 1 - 2 \frac{\tan^2 b}{\tan^2 c} = 1 - \frac{2}{4} \cdot \frac{16}{9} = \frac{1}{9}.$$

7055. (By Professor CAVALLIN, M.A.)—From an origin, within a closed convex contour, a perpendicular p , making an angle θ with some fixed direction, is drawn on the tangent at P , show that, if ρ be the radius of

curvature at P, and ψ some function of θ , then is in general

$$\int_0^{2\pi} \left(p\psi - \frac{dp}{d\theta} \frac{d\psi}{d\theta} \right) d\theta = \int_0^{2\pi} \rho\psi d\theta.$$

[As special cases this formula gives the length of the curve when $\psi = \text{constant}$, its area when $\psi = p$, and for $\psi = v^{-1}$, where v is the velocity at P of a point moving in the orbit, the time of a revolution of the point. The more general formula, which may also be easily proved, is

$$\int_0^{2\pi} \left(p\psi - \frac{dp}{d\theta} \frac{d\psi}{d\theta} \right) d\theta = \left(\psi \frac{dp}{d\theta} \right)_1 - \left(\psi \frac{dp}{d\theta} \right) + \int_{\theta_1}^{\theta_2} \rho\psi d\theta.]$$

Solution by J. HAMMOND, M.A.; G. G. MORRICE, M.A.; and others.

Integrating by parts, we have

$$\int_{\theta_1}^{\theta_2} \frac{dp}{d\theta} \frac{d\psi}{d\theta} d\theta = \left(\psi \frac{dp}{d\theta} \right)_2 - \left(\psi \frac{dp}{d\theta} \right)_1 - \int_{\theta_1}^{\theta_2} \psi \frac{d^2p}{d\theta^2} d\theta.$$

From this and the well-known result $p + \frac{d^2p}{d\theta^2} = \rho$, the theorem in question follows immediately.

With limits 0 and 2π , the two values of $\psi \frac{dp}{d\theta}$ are identical, and

$$\int_0^{2\pi} \left(p\psi - \frac{dp}{d\theta} \frac{d\psi}{d\theta} \right) d\theta = \int_0^{2\pi} \rho\psi d\theta.$$

6952. (By Professor TOWNSEND, F.R.S.)—Two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ being supposed to act in a common space; show that the complete locus of the entire system of points in the space for which their principal moments have similar or opposite directions, consists of three right lines, one situated at infinity in the space, and all three lying in planes parallel to the two central axes of the systems.

Solution by the PROPOSER; W. J. C. SHARP, M.A.; and others.

The two groups of six constituents of the two systems of forces, to any arbitrary triad of rectangular coordinate planes in the space, being $X_1Y_1Z_1L_1M_1N_1$ and $X_2Y_2Z_2L_2M_2N_2$ for $\Sigma(F_1)$ and $\Sigma(F_2)$ respectively; we have manifestly, for every point xyz of the entire system constituting the complete locus in question, the two equations

$$\frac{L_1 + Z_1y - Y_1z}{L_2 + Z_2y - Y_2z} = \frac{M_1 + X_1z - Z_1x}{M_2 + X_2z - Z_2x} = \frac{N_1 + Y_1x - X_1y}{N_2 + Y_2x - X_2y} \dots\dots\dots(1),$$

which, as is evident *a priori* they ought, are easily seen to be satisfied by all points xyz at infinity for which

$$x : y : z = (mX_1 + nX_2) : (mY_1 + nY_2) : (mZ_1 + nZ_2),$$

where m and n are any multiples, that is, by all points of the line of intersection with infinity of any plane parallel to the resultants R_1 and R_2 , or, which is the same thing, to the central axes I_1 and I_2 , of the two systems of forces; which line, as stated in the question, constitutes accordingly a part of the locus.

To determine the remainder of it. Denoting by k , a quantity to be subsequently determined, the common value of the three equivalents in the two equations (1), these two equations in xyz become replaced by the three in xyz and k ,

$$\left. \begin{aligned} (L_1 - kL_2) + (Z_1 - kZ_2)y - (Y_1 - kY_2)z &= 0 \\ (M_1 - kM_2) + (X_1 - kX_2)z - (Z_1 - kZ_2)x &= 0 \\ (N_1 - kN_2) + (Y_1 - kY_2)x + (X_1 - kX_2)y &= 0 \end{aligned} \right\} \dots\dots\dots(2),$$

which, representing manifestly for every satisfying value of k a triad of planes passing through a common line in the space, show consequently that the remainder of the locus consists of as many such lines as there are satisfying values of k , the number and magnitudes of which are obtained immediately as follows.

Multiplying the three equations (2) by $(X_1 - kX_2)$, $(Y_1 - kY_2)$, $(Z_1 - kZ_2)$, respectively, and adding; we see at once that x, y, z go out together, and that k is given by the residual equation

$$(X_1 - kX_2)(L_1 - kL_2) + (Y_1 - kY_2)(M_1 - kM_2) + (Z_1 - kZ_2)(N_1 - kN_2) = 0 \dots(3),$$

which shows that there are but two satisfying values of k , and that the remainder of the locus consists in consequence of the two lines of intersection of the two concurrent triads of planes represented by the equations (2), when k_1 and k_2 , the two roots of (3), are substituted in them successively for k .

That, as already shown for the third line at infinity, these two lines lie in two planes parallel to the resultants R_1 and R_2 of the two systems of forces, may be shown readily as follows. Multiplying the three equations (2) by X_2, Y_2, Z_2 respectively, and adding, we get at once, in xyz and k , the equation

$$\begin{aligned} (Y_1Z_2 - Z_1Y_2)x + (Z_1X_2 - X_1Z_2)y + (X_1Y_2 - Y_1X_2)z &= 0 \dots\dots(4), \\ + (L_1 - kL_2)X_2 + (M_1 - kM_2)Y_2 + (N_1 - kN_2)Z_2 & \end{aligned}$$

which, for the two values of k given by (3), representing evidently two planes perpendicular to the direction $(Y_1Z_2 - Z_1Y_2), (Z_1X_2 - X_1Z_2), (X_1Y_2 - Y_1X_2)$, and consequently parallel to the two directions $X_1Y_1Z_1$ and $X_2Y_2Z_2$; therefore, &c.

When the two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ in the space have each a single resultant; since then

$$L_1X_1 + M_1Y_1 + N_1Z_1 = 0 \quad \text{and} \quad L_2X_2 + M_2Y_2 + N_2Z_2 = 0,$$

the two roots of the quadratic (3) for k are respectively 0 and ∞ , and the two aforesaid lines of the locus are given consequently by the two groups of equations

$$\left. \begin{aligned} L_1 + Z_1y - Y_1z &= 0, \quad M_1 + X_1z - Z_1x = 0, \quad N_1 + Y_1x - X_1y = 0 \\ \text{and} \quad L_2 + Z_2y - Y_2z &= 0, \quad M_2 + X_2z - Z_2x = 0, \quad N_2 + Y_2x - X_2y = 0 \end{aligned} \right\} \dots(5),$$

which are those of the lines of action of the single resultants R_1 and R_2 of the two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ in the space; as it is manifest *a priori* they ought to be in that case.

7087. (By W. S. McCAY, M.A.)—Show that the problem, "To construct a triangle of given species and minimum area with its vertices on

the sides of a given triangle (Δ),” admits of two solutions Δ_1 and Δ_2 ; and if the perpendiculars to the sides of Δ at the vertices of Δ_1 and Δ_2 meet in P_1 and P_2 respectively, then P_1, P_2 are inverse points to the circumscribed circle of Δ , and $\Delta_1 : \Delta_2 = OP_1 : OP_2$, where O is the centre of the circle, and Δ_1, Δ_2 are the two minima areas.

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Solution by Professor WOLSTENHOLME, M.A.

I give the following construction for the two minimum triangles of given species to be inscribed in a given triangle (and at the same time for the two maximum triangles of the same species to be circumscribed to the same given triangle), but reserve the proof *in petto*, as it would involve considerable repetition of what I have given in the answer to the question of my own in the same number, and is mostly very elementary geometry which can readily be supplied.

Let ABC be the given triangle, and let A', B', C' be the angles of the triangle to be inscribed; on the sides BC, CA, AB describe segments of circles containing angles $\pi - A', \pi - B', \pi - C'$ (on the parts towards the opposite angles in each case); these will meet in a point S , and straight lines drawn through A, B, C at right angles to SA, SB, SC will form a maximum triangle of the required species; and if a conic be inscribed in the triangle ABC having S for one focus, and s be its second focus, the feet of the perpendiculars from s on the sides of the triangle ABC will be the corners of a minimum inscribed triangle of the same required species, and having its sides parallel to those of the maximum circumscribed triangle. The other maximum and minimum are formed in the same way by describing segments of circles containing angles A', B', C' which will meet in a point S' ; and s' is found from S' in just the same way as s from S . The areal coordinates of S, S' are to be found from the equations

$$(S) \dots X_1 (\cot A + \cot A') = Y_1 (\cot B + \cot B') = Z_1 (\cot C + \cot C'),$$

$$(S') \dots X_2 (\cot A - \cot A') = Y_2 (\cot B - \cot B') = Z_2 (\cot C - \cot C').$$

Hence the areal coordinates of s, s' are to be found from

$$(s) \dots \frac{x_1}{\frac{\sin A}{\sin A'} \sin (A + A')} = \frac{y_1}{\frac{\sin B}{\sin B'} \sin (B + B')} = \frac{z_1}{\frac{\sin C}{\sin C'} \sin (C + C')},$$

$$(s') \dots \frac{x_2}{\frac{\sin A}{\sin A'} \sin (A - A')} = \frac{y_2}{\frac{\sin B}{\sin B'} \sin (B - B')} = \frac{z_2}{\frac{\sin C}{\sin C'} \sin (C - C')};$$

which are obviously in the same straight line with

$$\frac{x}{\sin 2A} = \frac{y}{\sin 2B} = \frac{z}{\sin 2C},$$

the centre of the circle ABC ; and on substituting in the equation

$$a^2 (y_1 z_2 + y_2 z_1) + b^2 (z_1 x_2 + z_2 x_1) + c^2 (x_1 y_2 + x_2 y_1),$$

we, after some reduction, obtain an identity. Hence the two points s, s' are inverse points with respect to the circle ABC ; and if O be the centre of the circle, Δ_1, Δ_2 the areas of the minimum triangles,

$$2\Delta_1 = (R^2 - Os^2) \sin A \sin B \sin C, \quad 2\Delta_2 = (Os'^2 - R^2) \sin A \sin B \sin C,$$

so that

$$\Delta_1 : \Delta_2 = R^2 - Os^2 : Os'^2 - R^2,$$

or, since $R^2 = Os \cdot Os'$, $\Delta_1 : \Delta_2 = Os : Os'$.

There is an exceptional case when A', B', C' are respectively equal to

A, B, C, s' being then at infinity; so that only one minimum triangle similar to ABC can be inscribed (with the angles on the corresponding sides). The point S' may be any point on the circle ABC, and the corresponding maximum circumscribed triangles become both zero and indeterminate.

[The first result in question is an immediate consequence of the consideration that the inscription of triangles of given species (A'B'C') in a given triangle (ABC) reduces to finding a point whose distances from the vertices of the latter are connected by the equations $ar_1 : br_2 : cr_3 = a' : b' : c'$, the two points which satisfy this being inverses to the circumscribed circle, as determined at the intersections of circles orthogonal thereto.]

5740. (By Professor WOLSTENHOLME, M.A.)—If $a > b > c > d$, prove that

$$\int_b^a \frac{dx}{\left[(a-x)(x-b)(x-c)(x-d) \left\{ \frac{(a-x)(x-d)}{a-d} + \frac{(x-b)(x-c)}{b-c} \right\} \right]^{\frac{1}{2}}} \\ = \int_d^c \frac{dx}{\left[(a-x)(b-x)(c-x)(x-d) \left\{ \frac{(a-x)(x-d)}{a-d} + \frac{(b-x)(c-x)}{b-c} \right\} \right]^{\frac{1}{2}}}$$

[This question has been suggested by Quest. 5508.]

Solution by R. RAWSON; J. HAMMOND, M.A.; and others.

Consider the integral

$$u = \int_{\beta}^{\alpha} \frac{\left\{ \frac{(x-a)(x-b)(x-c)(x-d)}{(a-d)(b-c)} \right\}^{m-1} dx}{\left\{ \frac{(x-a)(x-d)}{a-d} - \frac{(x-b)(x-c)}{b-c} \right\}^{2m-1}} \dots\dots\dots(1),$$

which represents each side of the above equation when $m = \frac{3}{2}$.

$$\text{In (1) put } x = \frac{(a-b)c + a(b-c)y}{a-b + (b-c)y} = a - \frac{(a-b)(a-c)}{a-b + (b-c)y} \dots\dots\dots(2),$$

therefore

$$u = \frac{\left\{ \frac{(b-c)(c-a)}{(b-c)(a-a)} \right\}^{m-1}}{(-1)^{2m-1}} \int \frac{\left\{ \frac{(a-b)(c-a)}{(b-c)(a-a)} \right\}^{m-1} \left\{ y(1-y) \left(y + \frac{(a-b)(c-d)}{(a-d)(b-c)} \right) \right\}^{m-1} dy}{\left\{ \frac{(a-b)(c-d)}{(b-c)(b-a)} \right\} \left\{ y^2 + \frac{(a-b)(c-d)}{(a-d)(b-c)} \right\}^{2m-1}} \dots\dots\dots(3).$$

Let $u_1 = u$, when $\alpha = A$ and $\beta = B$; then we have

$$u_1 = \frac{\left\{ \frac{(b-c)(c-A)}{(b-c)(A-A)} \right\}^{m-1}}{(-1)^{2m-1}} \int \frac{\left\{ \frac{(a-b)(c-A)}{(b-c)(A-A)} \right\}^{m-1} \left\{ y(1-y)(y+r) \right\}^{m-1} dy}{\left\{ \frac{(a-b)(c-d)}{(b-c)(B-A)} \right\} \left\{ y^2 + r \right\}^{2m-1}} \dots\dots(4),$$

where

$$r = \frac{(a-b)(c-d)}{(a-d)(b-c)}.$$

Let $y = a$, when $x = A'$, and $y = B'$; then we have

$$\frac{(1-x)(x+r)}{(y^2+r)^{2n-1}} = \frac{(1-y)(y+r)}{(y^2+r)^{2n-1}} \cdot dy \dots (5)$$

Similarly (5) for the relation $y = -x$. Therefore

$$\frac{(1-x)(x+r)}{(x^2+r)^{2n-1}} = \frac{(1-x)(x+r)}{(x^2+r)^{2n-1}} \cdot dx$$

$$= \frac{(1-y)(y+r)}{(y^2+r)^{2n-1}} \cdot dy \dots (6)$$

Now, $y = -x$, provided the two following equations are satisfied, viz.,

$$\frac{(1-x)(x+r)}{(x^2+r)^{2n-1}} = \frac{(1-y)(y+r)}{(y^2+r)^{2n-1}} \dots (7)$$

Equation (7) is satisfied by $A = a$, $B = b$, $A' = x$, $B' = d$, and, if n is taken $\frac{1}{2}$, the truth of the equation in the question is manifest.

If $n = \frac{1}{2}$,

$$\int_a^b \frac{dx}{(1-x)(x+r)(x-d)(x-d)^2} = \int_a^b \frac{dx}{(1-x)(x-d)(x-d)(x-d)^2} \dots (8)$$

where A, B, A', B' must satisfy (7).

[Equation (8) is an interesting relation in elliptic functions, but it is, probably, to be found amongst the many equations which have been investigated by writers on elliptical functions.]

2294. [By Professor MANN, M.A.]—Three unequal homogeneous spheres being thrown into a given hemispherical bowl; determine their position when in equilibrium.

Solution by W. J. C. SMITH, M.A.; CHAMBERS A. SOWER; and others.

If r be the radius of the bowl, a, b, c those of the spheres, the position of equilibrium is that of a weightless triangle ABC , whose sides are $b+c$, $c+a$, and $a+b$, respectively; and which is loaded with weights proportional to a^3, b^3, c^3 at its angular points, and hung from the centre of the bowl by strings $r-a$, $r-b$, and $r-c$ in length. This condition is that G , the centre of gravity of the weights, lies vertically below O the centre of the bowl. If G_1 be the centre of gravity of the spheres at B and C ,

$$OG_1^2 = r^2 - 2r \frac{b^3 + c^3}{b^2 + c^2} + \left(\frac{b^4 - c^4}{b^2 + c^2} \right)^2, \quad AG_1^2 = a^2 + 2a \frac{b^3 + c^3}{b^2 + c^2} + \left(\frac{b^4 - c^4}{b^2 + c^2} \right)^2,$$

$$OG^2 = r^2 - 2r \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2} + \frac{a^4 + b^4 + c^4 - 2ab^3 - 2ba^3 - 2ac^3}{(a^2 + b^2 + c^2)^2},$$

which determines the depth of the centre of gravity; and with $OA = r - a$ and $AG = \frac{b^3 + c^3}{a^3 + b^3 + c^3} Ag_1$, the angle GOA and the inclinations of the other lines OB and OC to the vertical, may be similarly determined.

[The above values of Og_1 , Ag_1 and OG may be obtained by the repeated application of the formula $AD^2 = AB^2 \frac{CD}{BC} + AC^2 \frac{BD}{BC} - BD \cdot CD$, where ABC is a triangle, and D a point in BC .]

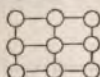
7039. (By T. C. SIMMONS, M.A.)—If two asymptotes of a cubic meet on the curve at a point of inflexion, prove that the third asymptote will pass through the same point.

7043. (By J. HAMMOND, M.A.)— A, B, C, D are four points on a cubic curve, BC meets the curve again in A' , CA in B' , AB in C' , AD in A'' , BD in B'' , CD in C'' ; prove that the lines $A'A''$, $B'B''$, $C'C''$ meet in a point which also lies on the curve.

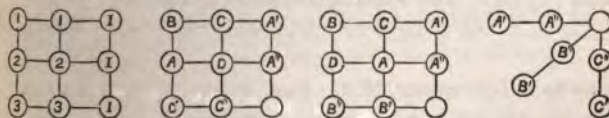
7058. (By J. J. WALKER, M.A.)—Show that Mr. HAMMOND's Quest. 7043 may be generalized thus:—If conics pass through four given points on a cubic, the chords of their remaining points of intersection with it meet in one point, which also lies on the cubic.

Solution by J. HAMMOND, M.A.; T. WOODCOCK, M.A.; and others.

The following is a convenient notation for the graphic representation of theorems of collineation on a cubic curve:—Points of a cubic curve are represented by circles connected by straight lines; when three points are collinear, their connectors are in the same direction, thus $\bigcirc - \bigcirc - \bigcirc$; when the names of the points are given, they may be inserted in the circles. The theorem that the connectors of collinear triads meet the curve again in a collinear triad, is represented thus



Let the three infinite points be denoted by ①, ②, ③, and the point of inflexion by I ; then, in 7039 and 7043, we have



Questions 7043 and 7058 are examples of Prof. SYLVESTER's theory of residuation (*Reprint*, Vol. xxxiv., p. 34), the second residue being a single point, in whatever way it is arrived at. These theorems include a

number of others, by supposing two or more of the points to coincide; for example, if a conic touch a cubic in two points, the chord joining the remaining points of intersection meets the cubic at the third intersection of the satellite of the chord of contact.

[Of Question 7058, which is also solved in SALMON's *Higher Curves*, 2nd ed., p. 131, Art. 154, Mr. WALKER gives the following solution:—

If $\alpha\beta, \gamma\delta$ are two pairs of lines passing through the four points, then any conic passing through the same points is of the form $\alpha\beta = k\gamma\delta$, and the cubic must be of the form $\alpha\beta\epsilon = \gamma\delta\zeta$, so that $\epsilon = k\zeta$ is the chord of the remaining intersections of the conic with the cubic. But this, whatever k is, passes through the point $\epsilon = 0, \zeta = 0$, which is also on the cubic.]

6902. (By G. F. WALKER, M.A.)—A series of quadrics $\frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} = 1$ is drawn through a fixed point (α, β, γ) : show that the locus of the centres of principal curvature at the fixed point is the surface

$$\alpha\gamma z (x-\alpha)^2 + \beta z x (y-\beta)^2 + \gamma x y (x-\gamma)^2 = 0.$$

Solution by MARY S. MEYER; T. WOODCOCK, B.A.; and others.

If (ξ, η, ζ) be a centre of principal curvature at $(\alpha\beta\gamma)$, we have

$$(\xi-\alpha) \frac{p}{\alpha} = (\eta-\beta) \frac{q}{\beta} = (\zeta-\gamma) \frac{r}{\gamma} = \frac{R}{K},$$

where $K^2 = \frac{\alpha^2}{p^2} + \frac{\beta^2}{q^2} + \frac{\gamma^2}{r^2}$, and R is (SALMON, Art. 295) given by

$$\frac{K^4}{R^2} - \frac{K}{R} \left[\left(\frac{1}{q} + \frac{1}{r} \right) \frac{\alpha^2}{p^2} + \dots + \dots \right] + \left(\frac{1}{qr} \frac{\alpha^2}{p^2} + \dots + \dots \right) = 0,$$

Hence we have $\left(1 + \frac{R}{K} \right) \left(1 + \frac{R}{Kr} \right) \frac{\alpha^2}{p^2} + \dots + \dots = 0$,

therefore $\frac{\eta\zeta}{\beta\gamma} (\xi-\alpha)^2 + \dots + \dots = 0$ is the required locus.

6974. (By A. McMURCHY, B.A.)—If $x+y+z=0$ and $a+b+c=0$, prove that $4(ax+by+cz)^3 - 3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) - 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x) = 54abcxyz$.

Solution by C. LEUDESORF, M.A.; PROF. FRISBY, M.A.; and others.

$$4(\Sigma ax)^2 - 3\Sigma a^2 \Sigma x^2 = (\Sigma ax)^2 - 3\Sigma (bx-cy)^2 = (\Sigma ax)^2 - [\Sigma a(y-z)]^2, \\ = \frac{1}{3}\Sigma a(y-z) = 4(bx+cy+az)(cx+ay+bz),$$

since $bz-cy=cx-az=ay-bx$; hence, if we write p for $(b-c)(c-a)(a-b)$,

and q for $(y-z)(z-x)(x-y)$, we have to prove that

$$2(ax+by+cz)(bx+cy+az)(cx+ay+bz) = pq + 27abcxyz.$$

Now, since $p = bc(c-b) + \&c.$ and $-3abc = bc(c+b) + \&c.$,

therefore $2\sum a^2b = -p - 3abc$, $2\sum ab^2 = p - 3abc$,

$$\therefore \sum a^2b \sum c^2y + \sum ab^2 \sum xy^2 = \frac{1}{4} [(p+3abc)(q+3xyz) + (p-3abc)(q-3xyz)] \\ = \frac{1}{2} (pq + 9abcxyz).$$

Multiply each side by 3, and then add to each $\sum a^3 \sum x^3 + 18abcxyz$, or its equivalent $27abcxyz$; then

$$(ax+by+cz)^3 + (bx+cy+az)^3 + (cx+ay+bz)^3 = \frac{1}{2} (3pq + 81abcxyz).$$

But the sum of the cubes on the left-hand side is equal to thrice the product of $ax+by+cz$, $bx+cy+az$, $cx+ay+bz$, since the sum of these quantities is zero; we have therefore what is required.

[Putting $a=0$, that is, substituting in sinister $a=0$, $b=-c$, $z=-(y+z)$, this sinister vanishes, therefore a is a factor thereof; and in the same way b , c , x , y , z are also factors; moreover, as the expression is homogeneous and of 6 dimensions, there is no other literal factor; and by putting the sinister $= N(abcxyz)$, and taking $a=b=x=y=1$, $c=z=-2$, we find the numerical value to be $N=54$.]

7137. (By R. KNOWLES, B.A., L.C.P.)—Prove that

$$S_1 \equiv \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \dots + \frac{(-1)^{n-1}}{2n(2n-1)} \text{ ad inf.} = \frac{1}{4}\pi - \frac{1}{2}\log 2;$$

$$S_2 \equiv \int_0^1 dx \log(1-x)^{\frac{1}{x}} = -\frac{1}{6}\pi^2.$$

Solution by the Rev. T. R. TERRY, M.A.; J. O'REGAN; and others.

$$S_1 = \int_0^1 dx \int_0^x (1-y^2+y^4\dots) dy = \int_0^1 \tan^{-1} x dx = \frac{1}{4}\pi - \frac{1}{2}\log 2.$$

$$S_2 = \int_0^1 = - \int_0^1 dx (1 + \frac{1}{2}x + \frac{1}{3}x^2\dots) = - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) = -\frac{1}{6}\pi^2.$$

7116. (By W. J. C. SHARP, M.A.)—If there be any number of centres of electric force (h_1, k_1, l_1) , (h_2, k_2, l_2) , &c., having charges ϵ_1, ϵ_2 , &c., respectively; prove that (1) the line of force at the point (x, y, z) whose distances from the centres are r_1, r_2 , &c., touches the vector

$$\sum \frac{\epsilon_1(x-h_1)}{r_1^3} \cdot i + \sum \frac{\epsilon_1(y-k_1)}{r_1^3} \cdot j + \sum \frac{\epsilon_1(z-l_1)}{r_1^3} \cdot k$$

at that point; and (2) this vector represents the force at the point in magnitude and direction.

Solution by J. W. SHARPE, M.A.; BELLE EASTON; and others.

The force exerted by ϵ_1 at the point p , upon a unit of electricity at p , is given by the vector

$$\pi (\rho - \rho_1) \times \frac{\epsilon_1}{r_1^3} = (\rho - \rho_1) \times \frac{\epsilon_1}{r_1^3} = \frac{\epsilon_1 (x - h_1)}{r_1^3} \cdot i + \frac{\epsilon_1 (y - k_1)}{r_1^3} \cdot j + \frac{\epsilon_1 (z - l_1)}{r_1^3} \cdot k;$$

whence, by the consideration of the whole system, the result follows.

6840. (By W. M. MEE, M.A.)—If from any point on a fixed ordinate of a parabola, three normals be drawn to the curve, prove that the centroid of the triangle whose corners are the three points where the normals cut the curve, is a fixed point on the axis.

Solution by E. BUSK, B.A.; J. M. REEVES, M.A.; and others.

Let $y^2 = 4mx$ and $x = a$ be the equations to the parabola and fixed ordinate, then (SALMON'S *Conics*, p. 194, ex. 9)

$$\begin{aligned} 4am &= 8m^2 + y_1^2 + y_2^2 + y_3^2 = 8m^2 + y_1^2 + y_1 y_3 + y_3^2 \\ &= 8m^2 + y_2^2 + y_2 y_3 + y_3^2 \dots\dots\dots (1, 2, 3). \end{aligned}$$

From (1) and (2), $y_2 + y_3 = -y_1 \dots\dots\dots (4)$,
and for the centroid

$$x_G = \frac{1}{3} (x_1 + x_2 + x_3) = \frac{1}{12m} [2 (y_2^2 + y_3^2) + 2y_2 y_3] = \frac{2}{3} (a - 2m), \text{ from (3);}$$

$$y_G = \frac{1}{3} (y_1 + y_2 + y_3) = 0, \text{ from (4); therefore, \&c.}$$

7114. (By J. HAMMOND, M.A.)—If A, B, C, D denote the first minors of the determinant

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \text{ its value is } Aa + Bb + Cc + Dd = pqrs,$$

where $s = a + b + c + d$, $p = d + a - b - c$,
 $q = d + b - c - a$, $r = d + c - b - a$;

prove that $s^3 = \frac{(D + A - B - C)(D + B - C - A)(D + C - A - B)}{(A + B + C + D)^2},$

with similar expressions for p^3, q^3, r^3 .

Solution by G. G. MORRICE, B.A.; SARAH MARKS, B.A.; and others.

If P, Q, R, S denote the corresponding functions of the large letters; viz., $D + A - B - C$, &c., we have

$$Pp = (D + A - B - C)(d + a - b - c) = aA + bB + cC + dD = pqrs,$$

for $aA + aB + aC + aD = 0$, $aB + aC + aD = 0$,
 and $aB + aC + aD = 0$,
 by a property of determinants. Similarly $Q_2 = 0$, $Q_3 = 0$,
 therefore $\frac{P_1 Q_1}{Q_1^2} = 0$, $\frac{Q_1 Q_2}{Q_2^2} = 0$, etc.

5644. (By W. J. C. FISHER, M.A.)—When the coefficient of a binary form (x, y) is a function of (x, y) , prove that the coefficient to the sum of whose $2n$ powers the function is reducible are the factors of the discriminant of (x, y) .

Solution by the Proposer.

By differentiation, if

$$(x, y, z, \dots, y)^{2n} = (\lambda_1 x + \mu_1 y)^{2n} + (\lambda_2 x + \mu_2 y)^{2n} + \dots + (\lambda_n x + \mu_n y)^{2n},$$

we have $ax + by = \lambda_1^{2n-1} (\lambda_1 x + \mu_1 y) + \lambda_2^{2n-1} (\lambda_2 x + \mu_2 y) + \dots$

$$bx + cy = \lambda_1^{2n-2} \mu_1 (\lambda_1 x + \mu_1 y) + \lambda_2^{2n-2} \mu_2 (\lambda_2 x + \mu_2 y) + \dots$$

$$cx + dy = \lambda_1^{2n-3} \mu_1^2 (\lambda_1 x + \mu_1 y) + \lambda_2^{2n-3} \mu_2^2 (\lambda_2 x + \mu_2 y) + \dots$$

therefore

$$\begin{vmatrix} ax + by & bx + cy & cx + dy \dots \\ bx + cy & cx + dy & dx + ey \dots \\ cx + dy & dx + ey & dx + ey \dots \end{vmatrix}$$

vanishes identically if $\lambda_1 x + \mu_1 y = 0$ or $\lambda_2 x + \mu_2 y = 0$, etc., which proves the proposition.

5843. (By T. R. TERRY, M.A.)—If the coordinates of the points A, B, C, D, be represented by (x_1, y_1, z_1) , and so on, prove that the equation to the tangent plane at A to the sphere which passes through A, B, C, D is

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ x - a_1 & y - a_2 & z - a_3 \end{vmatrix} = P [\mu_1 (x - a_1) + \mu_2 (y - a_2) + \mu_3 (z - a_3)] + Q [\lambda_1 (x - a_1) + \lambda_2 (y - a_2) + \lambda_3 (z - a_3)],$$

where $P = (\delta_1 - a_1) (c_1 - \delta_1) + (\delta_2 - a_2) (c_2 - \delta_2) + (\delta_3 - a_3) (c_3 - \delta_3)$,

$$Q = (d_1 - c_1) (a_1 - d_1) + \&c.$$

$$\lambda_1 = (\delta_2 - a_2) (c_3 - \delta_3) - (\delta_3 - a_3) (c_2 - \delta_2), \lambda_2 = \&c., \lambda_3 = \&c.,$$

$$\mu_1 = (d_3 - c_3) (a_2 - d_2) - (d_2 - c_2) (a_3 - d_3), \mu_2 = \&c., \mu_3 = \&c.$$

Solution by SARAH MARKS; the PROPOSER; and others.

In Sir W. R. HAMILTON's *Lectures on Quaternions*, Art. 329, it is proved that, considered as quaternions,

$$AB \cdot BC \cdot CD \cdot DA \cdot AR = AR \cdot DA \cdot CD \cdot BC \cdot AB,$$

where R is a point on the tangent plane at A. Hence

$$S(AB \cdot BC \cdot CD \cdot DA \cdot AR) = 0 \dots\dots\dots(1).$$

But $AB \cdot BC = -P + \lambda_1 i + \lambda_2 j + \lambda_3 k,$

and $CD \cdot DA = -Q + \mu_1 i + \mu_2 j + \mu_3 k,$

therefore, substituting in (1),

$$S(-P + \lambda_1 i + \dots)(-Q + \mu_1 i + \dots)[(x - a_1) i + \dots] = 0,$$

$$\text{therefore } - \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ x - a_1 & y - a_2 & z - a_3 \end{vmatrix} + P[\mu_1(x - a_1) + \dots] + Q[\lambda_1(x - a_1) + \dots] = 0,$$

which is the required result.

6812. (By Professor ΜΥΚΗΟΡΑΝΗΛΑΥ, M.A.)—Find the moment of inertia of (1) a circle, (2) a regular hexagon, in each case about an axis making an angle α with the plane of the figure.

Solution by E. BUSK, B.A.; E. RUTTER; and others.

Since the moments of inertia about more than one axis in the plane of the figure are the same, the moments about all axes in the plane are equal. If we take the axis of z perpendicular to the plane and passing through the centre, and any two rectangular axes in the plane as axes of x and y , we have

$$A = \Sigma m(y^2 + z^2) = \Sigma my^2 = \Sigma mx^2 = B, \quad C = \Sigma m(x^2 + y^2) = 2A;$$

therefore moment about an axis making an angle α with the plane

$$= A \cos^2 \alpha + C \sin^2 \alpha = A(\cos^2 \alpha + 2 \sin^2 \alpha) = A(1 + \sin^2 \alpha).$$

(1) In the case of the circle,

$$A = \Sigma m(x^2) = \frac{1}{2} \Sigma (mr^2) = \frac{1}{2} \mu \int_0^{2\pi} r^3 dr = \frac{\pi \mu a^4}{4} = M \cdot \frac{1}{4} a^2;$$

therefore required moment $= M \cdot \frac{1}{4} a^2 (1 + \sin^2 \alpha).$

(2) In the case of the hexagon,

$$AF = a = OF,$$

$$A = \Sigma (my^2),$$

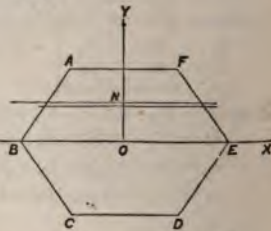
$$2\mu = \int_0^{a\frac{1}{2}\sqrt{3}} y^2 \left(a + \frac{a\sqrt{3}-2y}{a\sqrt{3}} \cdot a \right) dy$$

$$= 4\mu \int_0^{a\frac{1}{2}\sqrt{3}} \left(ay^2 - \frac{y^3}{\sqrt{3}} \right) dy$$

$$= 4\mu \left\{ a \cdot \frac{(a\frac{1}{2}\sqrt{3})^3}{3} - \frac{(a\frac{1}{2}\sqrt{3})^4}{4\sqrt{3}} \right\} = 4\mu a^4 \left(\frac{1}{8} \sqrt{3} \right)^3 \left\{ \frac{1}{3} - \frac{1}{8} \right\} = \frac{5\mu\sqrt{3}a^4}{16},$$

and $M = 6\mu \cdot a \cdot \frac{1}{2} a \sqrt{3}$; therefore $A = \frac{5}{24} M \cdot a^2$, therefore required

$$\text{moment} = M \frac{5a^2}{24} (1 + \sin^2 \alpha).$$



6079. (By S. TEWAY, B.A.)—The late SAMUEL BILLS has stated (*Reprint*, Vol. XXII., p. 71) that “any number can be resolved into three rational cubes.” Is this an established fact? Show that the equation $x^3 + y^3 + z^3 = 2\mu^3$ admits of a general solution; and thence find three numbers whose sum is unity, and such that, if each be taken from unity, the remainders shall be cubes.

Solution by the PROPOSER.

Let $x = aa_1$, $y = aa_2$, $z = aa_3$, $2\mu^3 = n$; then $a^{-3} = n^{-1}(a_1^3 + a_2^3 + a_3^3)$. Take $a_3 = p - s$, $a_1^3 + a_2^3 = nr^3 + s^3$ (see *Reprint*, Vol. XXV., p. 31), then

$$a^{-3} = r^3 + \frac{p^3}{n} - \frac{3ps^2}{n} + \frac{3ps^2}{n} = \left(r + \frac{ps^2}{nr^2}\right)^3, \text{ suppose;}$$

$$\text{then } p = \frac{3nr^3s}{nr^3 - s^3}, \quad a = \frac{1}{r} \cdot \frac{nr^3 - s^3}{nr^3 + 2s^3}, \quad a_3 = s \frac{2nr^3 + s^3}{nr^3 - s^3},$$

$$z = aa_3 = \frac{s}{r} \cdot \frac{2nr^3 + s^3}{nr^3 + 2s^3}.$$

We have now to solve $a_1^3 + a_2^3 = nr^3 + s^3 = 2\mu^3r^3 + s^3$.

Let $2\mu^3r^3 + s^3 = (\mu r + t)^3 + (\mu r - t)^3 = 2\mu^3r^3 + 6\mu r t^2$. Therefore $s^3 = 6\mu r t^2$. Take $r = 4\mu^2m^3$, and $t = 3n^2$. Then $s = 6\mu mn^2$. Therefore

$$a_1^3 + a_2^3 = (4\mu^3m^3 + 3n^3)^3 + (4\mu^3m^3 - 3n^3)^3.$$

We can therefore take $a_1 = 4\mu^3m^3 + 3n^3$, and $a_2 = 4\mu^3m^3 - 3n^3$. Hence we

$$\text{have } a = \frac{1}{8\mu^2m^3} \cdot \frac{16(\mu m)^6 - 27n^6}{8(\mu m)^6 + 27n^6}, \quad x = \frac{4\mu^3m^3 + 3n^3}{8\mu^2m^3} \cdot \frac{16(\mu m)^6 - 27n^6}{8(\mu m)^6 + 27n^6},$$

$$y = \frac{4\mu^3m^3 - 3n^3}{8\mu^2m^3} \cdot \frac{16(\mu m)^6 + 27n^6}{8(\mu m)^6 + 27n^6}, \quad z = \frac{3n^2}{4\mu m^2} \cdot \frac{32(\mu m)^6 + 27n^6}{8(\mu m)^6 + 27n^6}.$$

Let $\mu = 1$; then $x^3 + y^3 + z^3 = 2$. This is the value of $3a^3 - a$ when $a = 1$.

To satisfy the latter part of the problem, the ratio $\frac{s}{r}$ must be taken near its superior limit, which is $< \frac{2}{3}$. Take $m = 7$, $n = 4$; then we have

$$x = \frac{32667415}{45095239}, \quad y = \frac{40691280}{45095239}, \quad z = \frac{43298167}{45095239};$$

and the numbers are

$$1 - \left(\frac{32667415}{45095239}\right)^3 = \frac{34861358682288098698375}{91704802301923553136919},$$

$$1 - \left(\frac{40691280}{45095239}\right)^3 = \frac{67375818505225585152000}{91704802301923553136919},$$

$$1 - \left(\frac{43298167}{45095239}\right)^3 = \frac{81172427416333422423463}{91704802301923553136919}.$$

This subject is connected with the following general problem:—

“To find n numbers such that, if each be taken from the cube of their sum, the remainders shall be cubes” [which forms the first case of the general problem discussed by Dr. HART, under Quest. 4586, on pp. 66, 67 of Vol. xxvi. of our *Mathematical Reprints*].

Let $x_1, x_2, \dots, x_n$ be the numbers, a their sum; and suppose

$$a^3 - x_1 = a^3a_1^3, \quad a^3 - x_2 = a^3a_2^3, \quad \dots, \quad a^3 - x_n = a^3a_n^3;$$

then $x_1 = a^3(1 - a_1^3)$, $x_2 = a^3(1 - a_2^3)$, ..., $x_n = a^3(1 - a_n^3)$;

and, adding, $x_1 + x_2 + \dots + x_n = a = a^3(n - a_1^3 + a_2^3 - \dots - a_n^3)$.

Therefore

$$s^{-2} = s - a_1^2 - a_2^2 - \dots - a_n^2.$$

Let $s = 1$; then $s^{-2} = 1 - a_1^2$. Let $\frac{x}{y}$ be a value of s ; then

$$s^2 - s = s(x^2 - y^2)y^{-2};$$

hence $s(x^2 - y^2) \equiv s^3$, which can be shown to be impossible. Thus, the continued product of three numbers in arithmetical progression cannot be a cube.

Let $s = 2$; then $s^{-2} = 2 - a_1^2 - a_2^2$. Let $a_2 = 1 - a_1$; then we have

$$s^{-2} = 1 + 2a_1 - 2a_1^2 = (1 - sa_1)^2, \text{ suppose;}$$

therefore $a_1 = \frac{3+2s}{3+s^2}$; hence, putting $s = -1$, we have

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{2}, \quad s = \frac{1}{2}, \quad s_1 = \frac{27}{125}, \quad s_2 = \frac{63}{125}.$$

Let $s = 3$; then $s^{-2} = 3 - a_1^2 - a_2^2 - a_3^2$. If $\frac{x}{y}$ be a value of s , $s(3x^2 - y^2)$ must be the sum of three cubes; x and y not being necessarily prime to one another. A general solution of this case is still a *desideratum*. From a sieve, or table, consisting of the sums of three cube numbers, we find, after the manner of Dr. HART, the following triads of numbers:—

	x_1	x_2	x_3	CALCULATOR.
$\frac{7}{12}$	$\frac{19}{93}$	$\frac{817}{4116}$	$\frac{6155}{32928}$	TERAY.
$\frac{7}{12}$	$\frac{97}{612}$	$\frac{813}{4096}$	$\frac{2401}{12288}$	TERAY.
$\frac{7}{12}$	$\frac{335}{1728}$	$\frac{4459}{23328}$	$\frac{9253}{46656}$	TERAY.
$\frac{8}{13}$	$\frac{611}{2197}$	$\frac{11080}{59319}$	$\frac{11627}{59319}$	TERAY.
$\frac{9}{14}$	$\frac{13}{49}$	$\frac{17}{64}$	$\frac{351}{3136}$	HART.
$\frac{11}{18}$	$\frac{247}{1458}$	$\frac{8953}{39366}$	$\frac{8435}{39366}$	TERAY.
$\frac{19}{25}$	$\frac{1899}{15625}$	$\frac{4032}{15625}$	$\frac{4069}{15625}$	LENHART.
$\frac{18}{29}$	$\frac{4501}{24389}$	$\frac{5805}{24389}$	$\frac{4832}{24389}$	TERAY.
$\frac{18}{31}$	$\frac{45927}{238328}$	$\frac{5768}{36041}$	$\frac{46313}{238328}$	TERAY.
$\frac{27}{44}$	$\frac{13851}{85184}$	$\frac{18954}{85184}$	$\frac{19467}{85184}$	TERAY.
$\frac{27}{46}$	$\frac{853}{4232}$	$\frac{17955}{97336}$	$\frac{19558}{97336}$	TERAY.

Dr. HART's numbers are still the smallest that have been found. It was the object of the present inquiry to ascertain if smaller ones existed, but a searching scrutiny has failed to detect them. Unless some larger value of a produce results which admit of very considerable reductions, we must be content with Dr. HART's simple fractions. It is remarkable that the value $a = \frac{1}{\sqrt{2}}$ should produce three different triads of numbers; these are immediately obtained from $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}$.

7196. (By Professor TOWNSEND, F.R.S.)—Perpendiculars being supposed to fall from the centre of an ellipsoid abc upon the several osculating planes to the curve of intersection of any two confocal quadrics $a_1b_1c_1$ and $a_2b_2c_2$; investigate symmetrically the equation of the cone they determine, and show from it that the several sections of the cone by parallels to the principal planes of the system are the evolutes of conics.

Solution by the PROPOSER; Dr. CURTIS; and others.

The equation of the osculating plane at any point (x, y, z) of the curve of intersection of $a_1b_1c_1$ and $a_2b_2c_2$ (see SALMON'S *Geometry of Three Dimensions*, 4th ed., p. 329), being

$$a_1^{-2}a_2^{-2}a_3^2xx' + b_1^{-2}b_2^{-2}b_3^2yy' + c_1^{-2}c_2^{-2}c_3^2zz' = 1,$$

where $a_3b_3c_3$ is the third quadric of the system passing through (x, y, z) , therefore, for every point (ξ, η, ζ) on the corresponding perpendicular,

$$a_1^{-2}a_2^{-2}\xi : b_1^{-2}b_2^{-2}\eta : c_1^{-2}c_2^{-2}\zeta = a_3^2x : b_3^2y : c_3^2z,$$

or, since $(a^2 - b^2)(a^2 - c^2)x = a_1^2a_2^2a_3^2$, &c., &c. (see same, p. 142),

$$a_1^4a_2^4\xi : b_1^4b_2^4\eta : c_1^4c_2^4\zeta \\ = (a^2 - b^2)(a^2 - c^2)x : (b^2 - c^2)(b^2 - a^2)y : (c^2 - a^2)(c^2 - b^2)z;$$

substituting from which for xyz in the familiar relation

$$a_1^{-2}a_2^{-2}x^2 + b_1^{-2}b_2^{-2}y^2 + c_1^{-2}c_2^{-2}z^2 = 0,$$

(see same, page 143), there results immediately between ξ, η, ζ the relation

$$a_1^{\frac{2}{3}}a_2^{\frac{2}{3}}(b^2 - c^2)^{\frac{2}{3}}\xi^{\frac{2}{3}} + b_1^{\frac{2}{3}}b_2^{\frac{2}{3}}(c^2 - a^2)^{\frac{2}{3}}\eta^{\frac{2}{3}} + c_1^{\frac{2}{3}}c_2^{\frac{2}{3}}(a^2 - b^2)^{\frac{2}{3}}\zeta^{\frac{2}{3}} = 0,$$

which accordingly is the required equation, the form of which establishes at once the proposed property of the cone.

7180. (By A. McMURCHY, B.A.)—Prove that the whole volume of the solid bounded by the surface

$$\left(\frac{x}{a}\right)^{\frac{7}{3}} + \left(\frac{y}{b}\right)^{\frac{7}{3}} + \left(\frac{z}{c}\right)^{\frac{7}{3}} = 1, \text{ is } \left(\frac{5 \cdot 7}{9 \cdot 11 \cdot 13 \cdot 17 \cdot 19}\right) \cdot 4\pi abc.$$

Solution by W. H. BLYTHE, M.A.; SARAH MARKS; and others.

Consider the surface bounded by any surface of the form

$$\left(\frac{x}{a}\right)^{\frac{2}{n}} + \left(\frac{y}{b}\right)^{\frac{2}{n}} + \left(\frac{z}{c}\right)^{\frac{2}{n}} = 1,$$

where n is an odd positive integer.

Take an elementary section at a distance z from the origin, perpendicular to axis z , take

$$1 - \left(\frac{z}{c}\right)^{\frac{2}{n}} = \mu^2, \quad x = a\mu^n \cos^n \theta, \quad y = b\mu^n \sin^n \theta,$$

the area of the section is $4ab\mu^{2n} \cdot E$, where $E = n \int_0^{\frac{\pi}{2}} \cos^{n+1} \theta \sin^{n-1} \theta d\theta$

Next, integrating $4ab\mu^{2n} \cdot E \cdot dz$ twice from 0 to c , to find the volume of the whole solid, putting $z = c \sin^n \theta$, and therefore $\mu = \cos \theta$, we obtain

as final result, $4abcEB$, where $B = 2n \int_0^{\frac{\pi}{2}} \cos^{2n+1} \theta \sin^{n-1} \theta d\theta$. Using the

formulæ of reduction, we find that, when $n = 7$,

$$E = \frac{5 \cdot 7 \pi}{2^{12}}, \quad B = \frac{2^{12}}{9 \cdot 11 \cdot 13 \cdot 17 \cdot 19};$$

hence the volume required is as stated in the question.

[Otherwise:— $V = 8 \cdot \frac{abc}{(\frac{\pi}{2})^3} \cdot \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{3}{2})} =$ result given, obtained at once

by evaluating the Gamma Functions by means of

$$\Gamma(m) \Gamma(m + \frac{1}{2}) = \frac{\pi^{\frac{1}{2}}}{2^{2m-1}} \Gamma(2m)].$$

6971. (By G. S. CARR, B.A.)—If a number of voltaic cells differing both in electro-motive force and conductivity be joined “in multiple arc,” show that the difference of potentials of the electrodes of the battery, when connected by a wire, is equal to the sum of the current strengths of the separate cells divided by the sum of the conductivities of all parts of the circuit.

Solution by H. L. ORCHARD, M.A.; T. WOODCOCK, B.A.; and others.

If $e, e_1, \dots e_n$ be the different electro-motive forces, and $r, r_1, \dots r_n$ be the different resistances in the circuit, and C be the current strength,

$$\text{Ohm's Law gives } C = \frac{\sum e}{\sum r} = \frac{E}{R} \text{ say; therefore } E = \frac{C}{\frac{1}{R}} = \frac{C}{\sum \frac{1}{r}},$$

since the cells are in *multiple arc*, and $\sum \frac{1}{r}$ is the sum of the conductivities

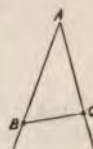
5490. (By S. TEBAÏ, B.A.)—AB, AC are two straight rods in a vertical plane, on which are two rings, B, C held in equilibrium by a flexible string passing through them, and having its ends fixed at A. If the equilibrium be slightly disturbed, find the time of a small oscillation.

Solution by the PROPOSER.

Let m, m' be the masses of B, C; T the tension of the string, l its length, α, α' the angles which AB, AC make with the vertical, $BAC = \Sigma$, $ABC = \phi$, $ACB = \phi'$, $AB = x$, $AC = y$, $BC = z$. Then, for the motions of B, C, we have

$$m \frac{d^2x}{dt^2} = gm \cos \alpha - 2T \cos^2 \frac{1}{2}\phi,$$

$$m' \frac{d^2y}{dt^2} = gm' \cos \alpha' - 2T \cos^2 \frac{1}{2}\phi';$$



$$\text{eliminating } T, m \cos^2 \frac{1}{2}\phi' \left(\frac{d^2x}{dt^2} - g \cos \alpha \right) - m \cos^2 \frac{1}{2}\phi \left(\frac{d^2y}{dt^2} - g \cos \alpha' \right) = 0.$$

By the geometry, $z \cos \phi = x - y \cos \Sigma$, $z \cos \phi' = y - x \cos \Sigma$;

therefore, eliminating ϕ, ϕ' , we have, as our first equation (1),

$$m(-x \cos \Sigma + y + z) \left(\frac{d^2x}{dt^2} - g \cos \alpha \right) - m'(x - y \cos \Sigma + z) \left(\frac{d^2y}{dt^2} - g \cos \alpha' \right) = 0.$$

Let a, b, c be the values of x, y, z , when there is equilibrium, and assume $x = a + x'$, $y = b + y'$, $z = c + z'$; x', y', z' being small quantities, such that

$$x' + y' + z' = 0 \dots\dots\dots(2).$$

Hence, neglecting small quantities of orders higher than the first, (1)

$$\begin{aligned} \text{becomes} \quad m p \frac{d^2x'}{dt^2} - m' q \frac{d^2y'}{dt^2} &= g m p \cos \alpha - g m' q \cos \alpha' \\ &+ g m \cos \alpha (-x' \cos \Sigma + y' + z') - g m' \cos \alpha' (x' - y' \cos \Sigma + z'); \end{aligned}$$

$$\text{putting} \quad l - 2a \cos^2 \frac{1}{2}\Sigma = p, \quad l - 2b \cos^2 \frac{1}{2}\Sigma = q.$$

When the system passes through the position of equilibrium,

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0,$$

and (1) gives $g m p \cos \alpha - g m' q \cos \alpha' = 0$. Therefore

$$\begin{aligned} m p \frac{d^2x'}{dt^2} - m' q \frac{d^2y'}{dt^2} &= g m \cos \alpha (-x' \cos \Sigma + y' + z') \\ &- g m' \cos \alpha' (x' - y' \cos \Sigma + z') \dots\dots\dots(3). \end{aligned}$$

Again, by the geometry, $z^2 = x^2 + y^2 - 2xy \cos \Sigma$,

$$\text{or} \quad (c + z')^2 = (a + x')^2 + (b + y')^2 - 2(a + x')(b + y') \cos \Sigma;$$

and observing that $c^2 = a^2 + b^2 - 2ab \cos \Sigma$, and neglecting small quantities of orders higher than the first,

$$cz' = (a - b \cos \Sigma) x' + (b - a \cos \Sigma) y' \dots\dots\dots(4).$$

$$\text{From (2, 4), } x' = \frac{p z'}{2(a - b) \cos^2 \frac{1}{2}\Sigma}, \quad y' = -\frac{q z'}{2(a - b) \cos^2 \frac{1}{2}\Sigma}.$$

Hence (3) becomes $\frac{d^2x}{dt^2} = 2g \cos^2 \frac{1}{2} \alpha \frac{pm \cos \alpha + qm' \cos \alpha'}{p^2m + q^2m'}$,

which shows that the system will oscillate in the same time as a simple pendulum whose length is

$$\frac{p^2m + q^2m'}{2 \cos^2 \frac{1}{2} \alpha (pm \cos \alpha + qm' \cos \alpha')}.$$

4981. (By S. TEBAY, B.A.)—A circular disc can revolve freely in a vertical plane about its axis, while a heavy particle descends in a curved linear groove between two given points in its plane; find the brachistochrone, and assign the limits of possibility.

Solution by the PROPOSER.

Let the upper point be the origin, m the mass of the particle, m' the mass of the disc, and θ the angle through which it has turned in time t . Then, by the principle of *vis viva*, we have

$$m \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} + k^2 m' \left(\frac{d\theta}{dt} \right)^2 = 2gmy;$$

or, putting $\frac{dy}{dx} = p$, $\frac{d\theta}{dx} = p'$, $k^2 \frac{m'}{m} = n^{-1}$,

$$t = \int \left(\frac{1+p^2+n^{-1}p'^2}{2gy} \right)^{\frac{1}{2}} dx, \text{ and } V = \left(\frac{1+p^2+n^{-1}p'^2}{y} \right)^{\frac{1}{2}}.$$

Since p, p' are independent, we have

$$\frac{d(V)}{dx} = Np + P \frac{dp}{dx} + P' \frac{dp'}{dx}, \quad N - \frac{d(P)}{dx} = 0, \quad \frac{d(P')}{dx} = 0.$$

Therefore $P' = \text{const.} = c'$, $\frac{d(V)}{dx} = \frac{d(Pp)}{dx} + c'dp'$,

and $V = Pp + c'p' + \frac{1}{c}.$

Now, $P = \frac{p}{\{y(1+p^2+n^{-1}p'^2)\}^{\frac{1}{2}}}$, $P' = \frac{n^{-1}p'}{\{y(1+p^2+n^{-1}p'^2)\}^{\frac{1}{2}}} = c'.$

Therefore $\frac{1+n^{-1}p'^2}{\{y(1+p^2+n^{-1}p'^2)\}^{\frac{1}{2}}} = c'p' + \frac{1}{c};$

and, by division, $p' = nc'c'$, and therefore $\theta = nc'c'x.$

From these equations, we find

$$\frac{dy}{dx} = \left(\frac{c^2}{y} - nc^2c'^2 - 1 \right)^{\frac{1}{2}} \dots\dots\dots (A).$$

Let a, b be the distances of the upper and lower points from the centre; α, β the angles which they make with the horizon; (x', y') the coordinates

of the point in the groove corresponding to (x, y) . The distance of the point xy from the centre is

$$\{(x - a \cos \alpha)^2 + (y - a \sin \alpha)^2\}^{\frac{1}{2}} = r,$$

which makes an angle $\sin^{-1} \left(\frac{y - a \sin \alpha}{r} \right)$ with the horizon. If the disc be turned back through the angle $\theta = ncc'x$, we have

$$x' = a \cos \alpha - r \cos \left\{ \sin^{-1} \left(\frac{y - a \sin \alpha}{r} \right) - ncc'x \right\},$$

$$y' = a \sin \alpha + r \sin \left\{ \sin^{-1} \left(\frac{y - a \sin \alpha}{r} \right) - ncc'x \right\}.$$

If we could eliminate xy between these equations and equation (A), we should obtain the equation to the brachystochrone required.

That the problem may be possible, it is necessary that

$$x = a \cos \alpha - b \cos (\beta + ncc'x), \quad y = a \sin \alpha + b \sin (\beta + ncc'x),$$

when the particle passes through the lower point. These equations combined with equation (A) furnish a relation between c, c' .

5710. (By E. W. SYMONS, M.A.)—If l_1, l_2, l_3, l_4 (of which l_4 is the greatest) be the semi-latus recta of the four conics circumscribing a triangle, each having one focus at the orthocentre, prove that

$$\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} > \frac{p_1 + p_2 + p_3}{\rho_2} > \frac{9}{l_4},$$

p_1, p_2, p_3 being the perpendiculars of the triangle, and ρ the radius of its self-conjugate circle.

Solution by J. A. KEALY, M.A.; the PROPOSER; and others.

If a, b, c be the sides of a Δ , $a^2 > (a - b + c)(a + b - c) \dots$,

therefore $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{\{(a + b - c)(a - b + c)\}^{\frac{1}{2}}} + \dots + \dots$;

but $\frac{2}{\{(a + b - c)(a - b + c)\}^{\frac{1}{2}}} < \frac{1}{a + b - c} + \frac{1}{a - b + c}$,

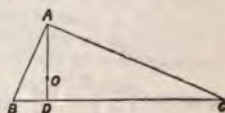
therefore $\frac{1}{\{(a + b - c)(a - b + c)\}^{\frac{1}{2}}} + \dots < \frac{1}{b + c - a} + \dots$,

hence, *a fortiori*, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{b + c - a} + \dots + \dots$

Again, $(a + b)(b + c)(c + a) > 8abc$,

or $(a + b + c)(bc + ca + ab) > 9abc$,

or $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{9}{a + b + c}$;



$$\therefore \frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > \frac{9}{a+b+c};$$

or, multiplying by 2Δ , we have, if r be radius of inscribed, and r_1, r_2, r_3 of escribed circumferences, $r_1 + r_2 + r_3 > p_1 + p_2 + p_3 > 9r$. Reciprocating this theorem with respect to the orthocentre, we shall get

$$\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} > \frac{1}{p'_1} + \frac{1}{\kappa_1} + \frac{1}{p'_2} + \frac{1}{\kappa_2} + \frac{1}{p'_3} + \frac{1}{\kappa_3} > \frac{9}{l^4},$$

($p'_1, p'_2, p'_3, \kappa_1, \kappa_2, \kappa_3$ being the distances of the orthocentre of the new Δ from its sides and vertices,)

$$\text{or } \frac{1}{l_1} + \dots > \frac{p'_1 + \kappa_1 + \dots}{p'_1 \cdot \kappa_1} > \frac{9}{l_1^4}, \quad \text{or } \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} > \frac{p_1 + p_2 + p_3}{\rho^3} > \frac{9}{l_1^4}.$$

7176. (By R. F. SCOTT, M.A.)—Through the middle point of the radius vector of the lemniscate $r^2 = a^2 \cos 2\theta$, a perpendicular is drawn to the radius vector; prove that the locus of the point in which this meets the normal to the lemniscate, at the extremity of the radius vector, is the hyperbola $4r^2 \cos 2\theta = a^2$.

Solution by Dr. CURTIS; J. O'REGAN; and others.

More generally, let the curve be $r^n = a^n \cos(n\theta)$, all else being as stated in the question, then, as in this class of curves the radius vector is inclined to the normal at an angle $= n\theta$ (see solution of Quest. 7148), if ρ and ϕ be the polar coordinates of the locus required, $\phi = (1-n)\theta$, and $\rho \cos n\theta = \frac{1}{2}r$, therefore $2^n \rho^n \cos^n n\theta = r^n = a^n \cos n\theta$, or

$$(2\rho)^n = a^n (\cos n\theta)^{1-n} = a^n \cos \left(\frac{n}{1-n} \phi \right),$$

$$\text{therefore } \rho^{\frac{n}{1-n}} = \left(\frac{1}{2}a \right)^{\frac{n}{1-n}} \cos \left(\frac{n}{n-1} \phi \right)^{1-n}.$$

If $n = 2$, we have $\rho^{-2} = \left(\frac{1}{2}a \right)^2 \cos(2\phi)$, or $4\rho^2 \cos 2\phi = a^2$.

6182. (By D. EDWARDES.)—The ends of a uniform rod of length $2a$ slide upon a smooth vertical circle (without inertia) of radius $\frac{2}{3}a\sqrt{3}$. If the system be set rotating about the fixed vertical diameter, prove that the inclination of the rod to the vertical at any time is given by the equation $\left(\frac{d\theta}{dt} \right)^2 = \frac{g\sqrt{3}}{a} (\sin \theta - \sin \alpha)$, α being the initial value of θ .

Solution by the PROPOSER; BELLE EASTON; and others.

The moment of inertia of rod in any position about the axis is $\frac{1}{3}a^2$. Let the lowest point of circle be origin, the fixed axis, the axis of z , and ϕ the angle described by the plane of the system in any time. Since the whole momentum about the axis is unchanged throughout the motion, we have $d\phi : dt = \text{constant}$. Hence, if (x, y, z) be a point on rod distant r from its centre, we have

$$x = \left(\frac{1}{3}a\sqrt{3}\cos\theta + r\sin\theta\right)\cos\phi, \quad y = \left(\frac{1}{3}a\sqrt{3}\cos\theta + r\sin\theta\right)\sin\phi, \\ z = r\cos\theta + a - \frac{1}{3}a\sqrt{3}\sin\theta.$$

The equation of energy is

$$\frac{1}{2} \int dr \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\} + g \int z dr = \text{constant},$$

or
$$a^2 \left(\frac{d\theta}{dt}\right)^2 + 3ag \left(1 - \frac{1}{\sqrt{3}}\sin\theta\right) = C;$$

and initially $\theta = \alpha$, $\frac{d\theta}{dt} = 0$, therefore

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{g\sqrt{3}}{a} (\sin\theta - \sin\alpha).$$

7109. (By J. J. WALKER, M.A.) — Show, from its fundamental property, that a principal axis (x) through any origin is determined by $\mathbf{V} \cdot \mathbf{a} \Sigma m \rho \mathbf{a} \rho = 0$, ρ being the vector of any element m of the body; and hence deduce some of the derived properties.

Solution by J. W. SHARPE, M.A.; Prof. MATZ, M.A.; and others.

The movement about any given vector λ is

$$\frac{\Sigma m T^2 \cdot \mathbf{V} \lambda \rho}{T^2 \lambda} = -\lambda^{-2} \Sigma m [\lambda^2 \rho^2 - (\mathbf{S} \lambda \rho)^2].$$

Now the function $\Sigma m (\lambda \rho^2 - \rho \mathbf{S} \lambda \rho)$ is a self-conjugate linear vector function in λ ; let it be denoted by $\mathbf{M} \phi \lambda$, where $\mathbf{M}$ is a scalar constant; then

$$\mathbf{M} \mathbf{S} \lambda \phi \lambda = \Sigma m [\lambda^2 \rho^2 - (\mathbf{S} \lambda \rho)^2] = (\text{moment of inertia about } \lambda) \times T^2 \lambda, \\ = \mathbf{L} \times T^2 \lambda, \text{ suppose; therefore } \mathbf{S} \lambda \phi \lambda = \frac{T^2 \lambda}{\mathbf{M}} \mathbf{L}.$$

Hence, if λ satisfy the equation $\mathbf{S} \lambda \phi \lambda = 1$, then $\mathbf{L} = \frac{\mathbf{M}}{T^2 \lambda}$,

where the moment of inertia about the vector λ is denoted by $\mathbf{L}$. Therefore the ellipsoid $\mathbf{S} \lambda \phi \lambda = 1$ is the momental ellipsoid at the origin, and its axes are given by the equation $\mathbf{V} \lambda \phi \lambda = 0$. This equation gives

$$\mathbf{V} \cdot \Sigma m \lambda \rho \mathbf{S} \lambda \rho = 0 \text{ or } \mathbf{V} \cdot \Sigma m (\lambda^2 \rho^2 - 2 \lambda \rho \mathbf{S} \lambda \rho) = 0,$$

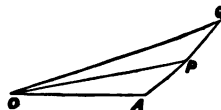
$$\text{or } \mathbf{V} \cdot \lambda \Sigma m (\lambda \rho^2 - 2 \rho \mathbf{S} \lambda \rho) = 0, \quad \text{that is, } \mathbf{V} \lambda \Sigma m \rho \lambda \rho = 0;$$

which is the equation required, having three vectors as resulting values for the variable λ , which give the directions of the three principal axes at the origin.

Let α denote a unit vector in the direction of one of the principal axes at O.

In the figure, let $OA = \alpha$, $OP = \rho$, and draw OQ in the plane of AOP , making

$$\angle QOP = \angle POA,$$



and such that OA , OP , OQ have their lengths in continued proportion; then $-\rho \alpha \rho = OQ$, and $T \cdot OQ = T^2 \rho$; and hence the equation for α gives $V \cdot \alpha \Sigma m (OQ) = 0$.

Now, let a particle of mass m be placed at Q ; then $\Sigma m (OQ)$ denotes the vector of the centre of gravity of this system of particles; and the equation for α is the condition that this vector be collinear with α . Now, let the origin be at the centre of gravity, then $\Sigma m \rho = 0$. The equation to the principal axes at any other point σ is $V \cdot \lambda \Sigma m (\rho - \sigma) \lambda (\rho - \sigma) = 0$,

or $V \cdot \lambda \Sigma m \rho \lambda \rho + V \cdot \lambda \Sigma m \sigma \lambda \sigma = 0$.

But $V \cdot \lambda \Sigma m \sigma \lambda \sigma = M \cdot V \lambda \sigma \lambda \sigma = 2MV \lambda \sigma S \lambda \sigma$,

where M = mass of body; therefore $V \cdot \lambda \Sigma m \rho \lambda \rho + 2MV \lambda \sigma S \lambda \sigma = 0$ is the equation to the principal axes at σ .

But the equation reduces to $V \cdot \lambda \Sigma m \rho \lambda \rho = 0$, if σ be in one of the principal axes at the centroid, for then $V \lambda \sigma = 0$, or if σ be in one of the principal planes, for then $S \lambda \sigma = 0$; and these results show that their principal axes at the centroid are principal axes at all points of their length, and that at a point in one of the principal planes at the centroid one of the principal axes is the normal to the plane at that point.

5894. (By Professor SEITZ, M.A.)—A circle is divided into two semi-circles, one of which is divided into two quadrants. If a point is taken at random in the surface of each quadrant, find the chance that the chord drawn through the points will be less than a line of given length.

Solution by the PROPOSER.

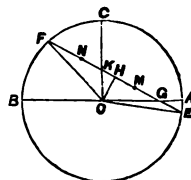
Let AOC and BOC be the two quadrants, M and N the random points, EF the chord through them, and O the centre of the circle. Draw OH perpendicular to EF .

Let 2θ be the angle formed by the radii to the extremities of a chord equal to the given line. Let $OA = r$, $KM = x$, $KN = y$, $KE = u$, $KF = u_1$, $KG = v$, $\angle EOH = \theta$, $\angle KOH = \phi$.

Then $u = r (\sin \theta + \cos \theta \tan \phi)$,

$$u_1 = r (\sin \theta - \cos \theta \tan \phi), \text{ and } v = r \cos \theta \sec \phi \operatorname{cosec} \phi.$$

An element of the circle at M is $r \sin \theta \, d\theta \, dx$, and at N it is $(x + y) \, dy \, d\phi$.



1. When $\beta > \frac{1}{2}\pi$, while $\theta < \frac{1}{2}\pi$, the limits of ϕ are 0 and θ , and doubled; those of x are 0 and u ; and those of y are 0 and u_1 . From $\theta = \frac{1}{2}\pi$ to $\theta = \beta$, the limits of ϕ are 0 and $\frac{1}{2}\pi - \theta$, and doubled, while those of x and y are the same as above; and then they are $\frac{1}{2}\pi - \theta$ and θ , and doubled, while those of x are 0 and v , and those of y are 0 and u_1 . Hence, since the whole number of ways the two points can be taken is $\frac{1}{16}\pi^2 r^4$, the required chance is

$$\begin{aligned}
 p &= \frac{32}{\pi^2 r^4} \int_0^{\frac{1}{2}\pi} \int_0^\theta \int_0^{u_1} r \sin \theta \, d\theta \, d\phi \, dx \, (x+y) \, dy \\
 &\quad + \frac{32}{\pi^2 r^4} \int_{\frac{1}{2}\pi}^\beta \left\{ \int_0^{\frac{1}{2}\pi-\theta} \int_0^u d\phi \, dx \, (x+y) \, dy + \int_{\frac{1}{2}\pi-\theta}^\theta \int_0^{u_1} d\phi \, dx \, (x+y) \, dy \right\} \\
 &\qquad \qquad \qquad \times r \sin \theta \, d\theta \\
 &= \frac{16}{\pi^2 r^3} \int_0^{\frac{1}{2}\pi} \int_0^\theta [(x+u_1)^2 - x^2] \sin \theta \, d\theta \, d\phi \, dx \\
 &\quad + \frac{16}{\pi^2 r^3} \int_{\frac{1}{2}\pi}^\beta \left\{ \int_0^{\frac{1}{2}\pi-\theta} [(x+u_1)^2 - x^2] d\phi \, dx + \int_{\frac{1}{2}\pi-\theta}^\theta [(x+u_1)^2 - x^2] d\phi \, dx \right\} \\
 &\qquad \qquad \qquad \times \sin \theta \, d\theta \\
 &= \frac{32}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^\theta (1 - \cos^2 \theta \sec^2 \phi) \sin^2 \theta \, d\theta \, d\phi \\
 &\quad + \frac{16}{\pi^2} \int_{\frac{1}{2}\pi}^\beta \left\{ \int_0^{\frac{1}{2}\pi-\theta} 2(1 - \cos^2 \theta \sec^2 \phi) \sin \theta \, d\phi \right. \\
 &\quad \left. + \int_{\frac{1}{2}\pi-\theta}^\theta [\sin \theta \cos^2 \theta (\operatorname{cosec}^2 \phi - \sec^2 \phi) - \cos \theta \cos 2\theta \operatorname{cosec} \phi \sec \phi] d\phi \right\} \\
 &\qquad \qquad \qquad \times \sin \theta \, d\theta \\
 &= \frac{32}{\pi^2} \int_0^{\frac{1}{2}\pi} (\theta - \sin \theta \cos \theta) \sin^2 \theta \, d\theta + \frac{16}{\pi^2} \int_{\frac{1}{2}\pi}^\beta [(\pi - 2\theta) \sin^2 \theta - 2 \sin \theta \cos^3 \theta \\
 &\quad - \sin 2\theta \cos 2\theta \log \tan \theta] d\theta \\
 &= \frac{1}{\pi^2} \{ 8\beta(\pi - \beta) - \pi^2 - 4(\pi - 2\beta) \sin 2\beta + 8 \cos^4 \beta - 4 \sin^2 2\beta \log \tan \beta \}.
 \end{aligned}$$

2. When $\beta < \frac{1}{2}\pi$, the required chance is found by changing the limits of θ in the first part of the above expression from 0 and $\frac{1}{2}\pi$ to 0 and β ;

therefore $p = \frac{32}{\pi^2} \int_0^\beta (\theta - \sin \theta \cos \theta) \sin^2 \theta \, d\theta = \frac{2}{\pi^2} (2\beta - \sin 2\beta)^2$.

When $\beta = \frac{1}{2}\pi$, both of the above results reduce to $p = \frac{1}{2} - \frac{2}{\pi} + \frac{2}{\pi^2}$.

7106. (By Professor HUDSON, M.A.)—Four heavy equal rough particles at the corners of a square are connected by smooth rigid weightless wires and rest on a horizontal table. A force is applied to the system along the

wire joining two of the particles, and is increased until the particles are on the point of motion. Obtain an equation to determine the point about which the system begins to turn.

Solution by J. W. SHARPE, M.A. ; CHARLOTTE A. SCOTT ; and others.

Let ABCD be the square at the corners of which the particles are placed. Draw EOF, DOG through the middle points of the sides. Let AD be the line of action of the force ; P the force, F the friction at each of the corners, Q the point about which the system begins to turn. Then at each corner there is a force F acting at right angles to the line joining the corner to Q.

Now (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , (x_4, y_4) denote the respective components of the force F at A, B, C, D ; the x components being parallel to OE, and the y components parallel to OG.

Then, by resolving and taking moments about Q,

$$y_1 + y_2 + y_3 + y_4 = P, \quad x_1 + x_2 + x_3 + x_4 = 0, \quad r_1 + r_2 + r_3 + r_4 = \frac{P}{F} p \dots (1, 2, 3),$$

where r_1, r_2, r_3, r_4 denote the distances of A, B, C, D respectively from Q, and P the perpendicular from Q on AD.

To obtain a solution of these equations, we must avail ourselves of some considerations of symmetry, together with the fact that the solution must be unique.

Now, if we take into consideration two positions of Q corresponding to each other in the two quarters of the square AEOG and DFOG, it will be at once evident that equation (2) would be satisfied for one position if satisfied for the other ; and that such would be the case also with equations (1) and (3), the values of the components of the frictions interchanging with one another in such a manner as not to change the value of $y_1 + y_2 + y_3 + y_4$, and to leave $x_1 + x_2 + x_3 + x_4$ zero for the 2nd position, if zero already for the 1st ; also, such a change in the position of Q would have no effect upon $r_1 + r_2 + r_3 + r_4$. But there exists no such symmetry with respect to the line EF. Hence the actual position of Q must be somewhere in the line DG.

Let it be at the point Q in OD, when $OQ = r$.

Let $2a$ = length of side of square,

$$r_1 = QA \text{ or } QD, \quad r_2 = QB \text{ or } QC ;$$

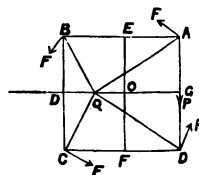
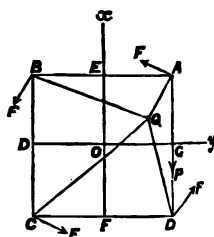
then the equations of equilibrium are [from (1) and (3), (2) being identically satisfied]

$$\frac{2(a+r)}{r_1} - \frac{2(a-r)}{r_2} = \frac{P}{F}, \quad 2r_1 + 2r_2 = \frac{P}{F}(a+r).$$

Eliminating $\frac{P}{F}$, we have $\frac{(a+r)}{r_1} - \frac{a^2-r^2}{r_2} = r_1 + r_2,$

where $r_1 = +[(a+r)^2 + a^2]^{\frac{1}{2}}, \quad r_2 = +[(a-r)^2 + a^2]^{\frac{1}{2}} ;$

and from this equation r must be found.



We shall now show that the equation has at least one real positive root. Writing it as follows—

$$(a+r)^2 r_2 - (a^2 - r^2) r_1 - r_1 r_2 (r_1 + r_2) = 0,$$

and denoting the expression on the left by $f(r)$, we have, when $\frac{a}{r}$ is very small,

$$\begin{aligned} \frac{f(r)}{r^3} &= \left(1 + \frac{2a}{r}\right) \left(1 - \frac{2a}{r}\right)^{\frac{1}{2}} + \left(1 + \frac{a}{r}\right)^{\frac{1}{2}} - \left(1 + \frac{2a}{r}\right)^{\frac{1}{2}} - \left(1 - \frac{2a}{r}\right)^{\frac{1}{2}} \\ &= \left(1 + \frac{2a}{r}\right) \left(1 - \frac{a}{r}\right) - \left(1 - \frac{a}{r}\right) = \frac{2a}{r}; \end{aligned}$$

therefore $f(r) = 2ar^2$ when r is very large compared with a ; therefore $f(\infty)$ and $f(-\infty)$ are both positive.

Also $f(0) = a^3(\sqrt{2} - \sqrt{2} - 4\sqrt{2}) = -4\sqrt{2}a^3$;

and therefore the equation has at least one positive real root.

6680. (By Professor МУКНОРАДНУАУ, M.A.)—Trisect a given triangle by straight lines at right angles to the base.

Solution by Rev. T. C. SIMMONS, M.A.; E. RUTTER; and others.

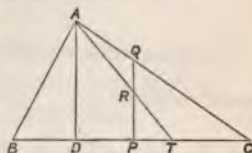
Let T be one of the points of trisection of the base, and suppose it to lie between D and C . Take CP a mean proportional between CT and CD ; then the perpendicular PQ will be one of the required lines. For, let it meet AT in R . Then since

$$CT : CP = CP : CD = CQ : CA,$$

therefore QT is parallel to AP ; whence it easily follows that $\triangle ARQ = \triangle RPT$, and $\therefore \triangle CQP = \triangle CAT = \frac{1}{3}\triangle CAB$.

If t be the other part of trisection of BC , then, if it lie between C and D , p the foot of the other perpendicular may be found from the relation $cp^2 = ct \cdot CD$. But if t lies between D and B we must use the relation $Bp^2 = Bt \cdot BD$.

Thus the triangle is completely trisected.



4622. (By S. TERAY.)—A fixed pulley rests on a smooth horizontal plane, and a string is fastened to its rim, and a weight (P) to the other end of the string, which lies on the plane, the string being tangential to the pulley. If the weight receives a given impulse perpendicular to the

string, determine the motion, and show that there is an epoch at which the string ceases to coil round the pulley. Assign the subsequent motion of P.

Solution by the PROPOSER.

Let CA be the initial radius, which in the time t has revolved into the position CA', while the string BP is tangential to the pulley at B. Let x, y be the coordinates of P, CA = a , $\angle ACA' = \theta$, $\angle ACB = \phi$, l the length of the string, BP = r , m the mass of the pulley, m' the mass of P, and v its initial velocity. Then, by the conservation of areas,

$$k^2 m \left(\frac{d\theta}{dt} \right) + m' \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = lm'v;$$

and, by the conservation of vis viva,

$$k^2 m \left(\frac{d\theta}{dt} \right)^2 + m' \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} = m'v^2.$$

Now $x = a \cos \phi - r \sin \phi$, $y = a \sin \phi + r \cos \phi$, $l = r + a(\phi - \theta)$,

$$\frac{dx}{dt} = -r \cos \phi \frac{d\phi}{dt} - a \sin \phi \frac{d\theta}{dt}, \quad \frac{dy}{dt} = -r \sin \phi \frac{d\phi}{dt} + a \cos \phi \frac{d\theta}{dt};$$

therefore

$$a^2 \left(\frac{1}{2}m + m' \right) \frac{d\theta}{dt} + m'r^2 \frac{d\phi}{dt} = lm'v,$$

$$a^2 \left(\frac{1}{2}m + m' \right) \left(\frac{d\theta}{dt} \right)^2 + m'r^2 \left(\frac{d\phi}{dt} \right)^2 = m'v^2.$$

The string will cease to coil round the pulley when $\frac{d\theta}{dt} = \frac{d\phi}{dt}$, and this condition gives

$$r'^2 = l^2 - a^2 \left(\frac{1}{2} \frac{m}{m'} + 1 \right).$$

Let $h^2 = a^2 \left(\frac{1}{2} \frac{m}{m'} + 1 \right)$, $R^2 = r^2 + h^2 - l^2$, and we have

$$\frac{1}{a} \frac{dr}{d\phi} = - \frac{lrR - hR^2}{h(l^2 - h^2)}, \text{ whence } a\phi = 2h \left\{ \tan^{-1} \frac{r-R}{l-h} - \frac{\pi}{4} \right\}.$$

Also,

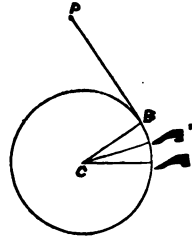
$$a\theta = 2h \left\{ \tan^{-1} \frac{r-R}{l-h} - \frac{\pi}{4} \right\} - l + r.$$

Again, $\frac{d\phi}{dt} = \frac{v}{r} \cdot \frac{lr + hR}{h^2 + r^2}$, therefore $\frac{1}{a} \frac{dr}{dt} = - \frac{v}{h} \cdot \frac{R}{r}$;

and $at = \frac{h}{v} (h - R)$; hence, when the motion becomes uniform,

$$a\phi' = 2h \left\{ \tan^{-1} \left(\frac{l+h}{l-h} \right)^{\frac{1}{2}} - \frac{\pi}{4} \right\},$$

$$a\theta = 2h \left\{ \tan^{-1} \left(\frac{l+h}{l-h} \right)^{\frac{1}{2}} - \frac{\pi}{4} \right\} - l + (l^2 - h^2)^{\frac{1}{2}}, \quad at' = \frac{h^2}{v};$$



At this epoch $\frac{d\theta}{dt} = \frac{d\phi}{dt} = \frac{v}{l}$; and, if V be the velocity of P ,

$$V^2 = \frac{v^2}{l^2} \left(l^2 - \frac{1}{2} a^2 \frac{m}{m'} \right).$$

Let $x' = a \cos \phi' - r' \sin \phi'$, $y' = a \sin \phi' + r' \cos \phi'$,

$$\mu = \left(\frac{dy}{dx} \right) = \frac{r' \sin \phi' - a \cos \phi'}{r' \cos \phi' + a \sin \phi'} = -\frac{x'}{y'};$$

then the path of P is the straight line

$$y - y' = \mu (x - x'), \text{ or } x'x + y'y = a^2 + r'^2,$$

that is, perpendicular to CP .

7142. (By Capt. P. A. MACMAHON.)—Deal n^n cards into n packs, one card from the top to each pack in succession. Let a second person choose a card, and name the pack in which it is. Gather up the cards so that the named pack may be the m_1^{th} pack from the top. Repeat the above, placing the named pack m_2^{th} from the top, and so on n times in succession. Find $m_1, m_2, m_3 \dots m_n$, so that the card chosen (which is unknown to the dealer) may be the x^{th} card from the top at the end of the n^{th} deal, x being any number from 1 to n^n inclusive.

I. *Solution by the PROPOSER; J. S. JENKINS, M.A.; and others.*

Put $x + n^{n-1} + n^{n-2} + \dots + n^2 + n = Q_1 n + R_1,$

$$Q_1 = Q_2 n + R_2, \quad Q_2 = Q_3 n + R_3, \quad \dots, \quad Q_{n-1} = 0 \cdot n + R_n,$$

where $R_1, R_2 \dots$ may be numbers from 1 to n inclusive, but not zero.

Then will $m_1, m_2, \dots m_n = R_1, R_2, \dots R_n$ respectively.

II. *Solution by the Rev. T. R. TERRY, M.A.; CHRISTINE LADD, B.A.; and others.*

As the directions about dealing are not sufficiently clear, we may suppose that, when the cards are being dealt, they are held face downwards, but that, on being dealt onto the heaps, each card is turned face upwards. Thus, when preparing for a new deal, the dealer turns each heap face downwards, placing the named heap in the particular position assigned by the law. Required to find this law.

If at any stage of the proceedings the selected card is in the $(qn+r)^{\text{th}}$ place in its heap, and if this heap is then placed p^{th} from the top, it is clear that after the next deal the card will be in the $[(p-1)n^{n-2} + (q+1)]^{\text{th}}$ place in the heap into which it has been dealt. Hence it follows at once that

$$x-1 = (m_1-1) + (m_2-1)n + \dots + (m_n-1)n^{n-1}.$$

Thus the simple rule is: Express $(x-1)$ in the scale of n , let $r_1, r_2 \dots r_n$ be the digits found in order, then $m_p = r_p + 1$, where p is any integer not greater than n .

6519. (By W. J. C. SMITH, M.A.)—With the notation adopted in Question 5433, prove that

$$\frac{x^{p^i-1}-1}{(x-1)(x^2-1)\dots(x^{p^{i-1}}-1)} = \frac{x^{p^{i-1}-1}-1}{p-1} X_{ij} + \frac{x^{p^i-1}-1}{p-1} X_{ij-1} \\ + p \frac{x^{p^{i-1}-1}-1}{p-1} X_{ij-2} + p^2 \frac{x^{p^{i-2}-1}-1}{p-1} X_{ij-3} + p^3 \frac{x^{p^{i-3}-1}-1}{p-1} X_{ij-4} + \&c.$$

The Note in Question 5433, from the words "of course, when ij is even," applies to this Question as well.

Solution by the Proposer.

$$\text{If } \frac{1-x^{p^i-1}}{(1-x)(1-x^2)\dots(1-x^{p^{i-1}})} \equiv B_0 + B_1 x + B_2 x^2 + \dots + B_{ij-1} x^{ij-1},$$

then, since $B_{ij-1} = -B_0$, we have

$$\frac{1-x^{p^i-1}}{(1-x)(1-x^2)\dots(1-x^{p^{i-1}})} \\ \equiv B_0 \frac{1-x^{p^i-1}}{1-x} + B_1 x \frac{1-x^{p^i-1}}{1-x} + B_2 x^2 \frac{1-x^{p^i-1}}{1-x} + \&c.$$

But B_0 is the number of covariants of order j in the coefficients and weights of source possessed by a binary -ic, and $B_r = X_{ij-r}$ (SALMON'S *Higher Algebra*, pp. 115 and 272):

$$\text{therefore } \frac{(x^{p^i-1}-1)(x^{p^{i-1}-1}-1)\dots(x^{p-j}-1)}{(x-1)(x^2-1)\dots(x^j-1)} \\ = X_{ij} \frac{x^{p^i-1}-1}{x-1} + X_{ij-2} x \frac{x^{p^i-1}-1}{x-1} + X_{ij-4} x^2 \frac{x^{p^i-1}-1}{x-1} + \&c.,$$

which (the superfluous terms vanishing of themselves) gives the correct result

$$\frac{(p^{i-1}-1)(p^{i-2}-1)\dots(p^{i-j}-1)}{(p-1)(p^2-1)\dots(p^j-1)} = X_{ij} \frac{p^{p^i-1}-1}{p-1} + X_{ij-2} p \frac{p^{p^i-1}-1}{p-1} + \&c.,$$

for all values of p .

It is evident that X_{ij} is always unity, and the other X 's can always be determined by giving a sufficient number of values to p .

[Professor SYLVESTER'S result in 5493 follows by making $p = 1$, and adding the results when ij is even and when it is odd.]

7118. (By D. EDWARDS.)—Prove that (1) the area of the triangle formed by joining the feet of the normals drawn from a point (X, Y) to the parabola $y^2 = 4ax$, is $\Delta = a^{\frac{1}{2}} [4(X-2a)^3 - 27aY^2]^{\frac{1}{2}}$; (2) if the triangle is right-angled, $\Delta = a^{\frac{1}{2}} (4X^3 - 24aX^2 + 21a^2X + 49a^3)^{\frac{1}{2}}$, and the hypotenuse cuts the axis at a distance from the vertex equal to one-fourth of the *latus rectum*.

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$$2. \quad \int_0^1 \frac{\phi \left(\sin^2 \alpha + \frac{\cos^2 \alpha}{x} \right)}{(x-1)(x+\cos^2 \alpha)} dx = \int_{-\infty}^1 \frac{\phi(y)}{(y-1)(y+\cos^2 \alpha)} dy, \\ \left(\text{if } x = \frac{\cos^2 \alpha}{y - \sin^2 \alpha} \right),$$

$$\text{therefore} \quad \int_1^{\infty} \frac{\phi(x)}{(x-1)(x+\cos^2 \alpha)} dx + \int_0^1 \frac{\phi(x)}{(x-1)(x+\cos^2 \alpha)} dx = 0,$$

$$\text{therefore} \quad \int_0^{\infty} \frac{\phi(x)}{(x-1)(x+\cos^2 \alpha)} dx = 0.$$

$$\text{Also} \quad \int_0^1 \frac{\phi \left(\sin^2 \alpha + \frac{\cos^2 \alpha}{x} \right)}{(x-1)^2} dx = \cos^2 \alpha \int_1^{\infty} \frac{\phi(y)}{(y-1)^2} dy \left(\text{if } x = \frac{\cos^2 \alpha}{y - \sin^2 \alpha} \right),$$

$$\text{or} \quad (1 + \cos^2 \alpha) \int_0^1 \frac{\phi(x)}{(x-1)^2} dx = \cos^2 \alpha \int_0^{\infty} \frac{\phi(x)}{(x-1)^2} dx.$$

6250. (By Professor SEITZ, M.A.)—A, B, C are random points within a hemisphere; find the chance that the plane through A, B, C does not intersect the base of the hemisphere.

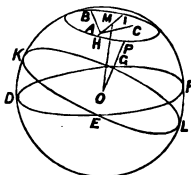
Solution by the PROPOSER; Professor NASH, M.A.; and others.

Let DEFG be the base of the hemisphere, O its centre, OM the perpendicular on the plane through A, B, C, HAI a diameter of the section made by this plane, OP a line such that AB is parallel to the plane MOP, and KELG a great circle of the complete sphere, perpendicular to OP.

Now, instead of confining the points A, B, C to the given hemisphere, we may confine them to the hemisphere whose base is KELG, and find the chance that the plane through them does not intersect the base KELG. The result will evidently be the same as that required in the question. Let OH = r , MA = x , AB = y , AC = z , $\angle HOM = \theta$, $\angle BAM = \phi$, $\angle CAM = \psi$, $\angle MOP = \mu$, and the angle the plane MOP makes with a fixed plane through OP = ω .

Then an element of the sphere at A is $r \sin \theta d\theta \cdot 2\pi x dx$, at B it is $y^2 dy d\phi d\mu$, and at C it is $\sin(\phi + \psi) \sin \mu z^2 dz d\psi d\omega$. The limits of θ are 0 and $\frac{1}{2}\pi$; of x , 0 and $2r \sin \theta = x'$, and tripled; of ϕ , $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ , $-\phi$ and $\frac{1}{2}\pi$; and doubled; of μ , 0 and $\frac{1}{2}\pi - \theta$; of ω , 0 and 2π ; of y , 0 and $2x \cos \phi = y'$; and of z , 0 and $2x \cos \psi = z'$. Hence, since the whole number of ways the three points can be taken within a hemisphere is $(\frac{1}{2}\pi r^3)^3$, the required chance is

$$\frac{6}{(\frac{1}{2}\pi r^3)^3} \int_0^{\frac{1}{2}\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi - \theta} \int_0^{\frac{1}{2}\pi - \theta} \int_0^{2\pi} \int_0^{y'} \int_0^{z'} r \sin \theta d\theta \cdot 2\pi x dx d\phi \sin(\phi + \psi) d\psi \\ \times \sin \mu d\mu d\omega y^2 dy z^2 dz$$



$$\begin{aligned}
&= \frac{108}{\pi^2 \gamma^3} \int_0^{1\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{1\pi} \int_{-\phi}^{1\pi} \int_0^{1\pi-\theta} \int_0^{2\pi} \int_0^{y'} \sin \theta \, d\theta \, x^4 \, dx \, d\phi \, \cos^3 \psi \sin(\phi + \psi) \, d\psi \\
&\quad \times \sin \mu \, d\mu \, d\omega \, y^2 \, dy \\
&= \frac{288}{\pi^2 \gamma^3} \int_0^{1\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{1\pi} \int_{-\phi}^{1\pi} \int_0^{1\pi-\theta} \int_0^{2\pi} \sin \theta \, d\theta \, x^7 \, dx \, \cos^3 \phi \, d\phi \, \cos^3 \psi \sin(\phi + \psi) \, d\psi \\
&\quad \times \sin \mu \, d\mu \, d\omega \\
&= \frac{576}{\pi^2 \gamma^3} \int_0^{1\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{1\pi} \int_{-\phi}^{1\pi} \int_0^{1\pi-\theta} \sin \theta \, d\theta \, x^7 \, dx \, \cos^3 \phi \, d\phi \, \cos^3 \psi \sin(\phi + \psi) \, d\psi \\
&\quad \times \sin \mu \, d\mu \\
&= \frac{576}{\pi^2 \gamma^3} \int_0^{1\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{1\pi} \int_{-\phi}^{1\pi} (1 - \sin \theta) \sin \theta \, d\theta \, x^7 \, dx \, \cos^3 \phi \, d\phi \, \cos^3 \psi \sin(\phi + \psi) \, d\psi \\
&= \frac{72}{\pi^2 \gamma^3} \int_0^{1\pi} \int_0^{x'} \int_{-\frac{1}{2}\pi}^{1\pi} [3(\frac{1}{2}\pi + \phi) \sin \phi + 2 \cos \phi + \sin^2 \phi \cos \phi] (1 - \sin \theta) \sin \theta \, d\theta \\
&\quad \times x^7 \, dx \, \cos^3 \phi \, d\phi \\
&= \frac{315}{4\gamma^3} \int_0^{1\pi} \int_0^{x'} (1 - \sin \theta) \sin \theta \, d\theta \, x^7 \, dx = \frac{315}{32} \int_0^{1\pi} (1 - \sin \theta) \sin^3 \theta \, d\theta \\
&\quad = 4 - 6\pi \left(\frac{63}{128} \right)^2.
\end{aligned}$$

7041. (By W. J. C. SHARP, M.A.)—Find the form of the catenary in which an extensible string, uniform when unstretched, will hang on the centre of gravity.

Solution by the PROPOSER.

Generally, if s be the arc of any catenary from a fixed point to any point P, and θ the angle between the tangents at these points; and tds , nds be the tangential and normal forces on ds situated at P, and T the tension at

that point; $\frac{dT}{ds} = t$ and $T \frac{d\theta}{ds} = u$;

and for gravitation catenaries, if the tangent at the initial point be horizontal, and ω be the weight of the string at P per unit of length;

$$= -\omega \sin \theta, \quad n = -\omega \cos \theta, \quad \text{and therefore } T = -a \sec \theta,$$

and
$$\frac{ds}{d\theta} = \frac{a}{\omega} \sec^2 \theta.$$

In the question, ω is a variable which is connected with ω_0 , the weight of the unstretched string per unit of length, by the equation

$$\omega = \frac{\omega_0}{1 + \frac{T}{\lambda}} = \frac{\omega_0}{1 + \frac{a}{\lambda} \sec \theta};$$

therefore
$$\frac{ds}{d\theta} = \frac{a}{\omega_0} \left(1 + \frac{a}{\lambda} \sec \theta \right) \sec^2 \theta.$$

$$s = \frac{a}{\omega_0} \tan \theta + \frac{a^2}{2\omega_0 \lambda} \sec \theta \tan \theta + \frac{a^2}{2\omega_0 \lambda} \log \frac{1 + \sin \theta}{\cos \theta} \dots\dots\dots(1),$$

no constant being added, since s and θ vanish together.



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VOL. XXXIX.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1883.

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CONTENTS.

Solved Questions.

1100. (By the Editor.)—If a straight line be divided at random into three parts, show that the chance (1) that a triangle of any kind can be formed with the three parts is $\frac{1}{4}$; and (2) that an acute-angled triangle can be formed by the three parts is $3 \log 2 - 2$ 37

3083 & 5143. (By Dr. James Matteson.)—Find a common value of x that will make $x^2 \pm ax = \square$, $x^2 \pm bx = \square$, $x^2 \pm cx = \square$, &c., *ad inf.*; and again $a^2x^2 \pm ax = \square$, $b^2x^2 \pm bx = \square$, $c^2x^2 \pm cx = \square$, &c., *ad. inf.* 47

3267. (By Professor Siverly, M.A.)—Determine the axes of the maximum ellipse inscribed in the quadrant of a given ellipse 77

4387. (By Professor Cayley, F.R.S.)—Using the term "Cassinian" to denote a bi-circular quartic having four foci in a right line; show that the equation of a Cassinian having for its four foci the points $x = a$, $x = b$, $x = c$, $x = d$ on the axis of x , may be written in the four equivalent forms

$$\begin{pmatrix} \tau(d-c), & \sigma(b-d), & \rho(c-b) \\ \tau(c-d), & \rho(d-a), & \sigma(a-c) \\ \sigma(d-b), & \rho(a-d), & \tau(b-a) \\ \rho(b-c), & \sigma(c-a), & \tau(a-b) \end{pmatrix} (A^4, B^4, C^4, D^4) = 0,$$

that is, $\tau(d-c)B^4 + \sigma(b-d)C^4 + \rho(c-b)D^4 = 0$,
 $\tau(c-d)A^4 + \rho(d-a)C^4 + \sigma(a-c)D^4 = 0$,
 &c., &c.,

where A^4, B^4, C^4, D^4 are the distances from the four foci respectively, and the parameters ρ, σ, τ are connected by the equation

$$\rho^2(a-d)(b-c) + \sigma^2(b-d)(c-a) + \tau^2(c-d)(a-b) = 0.$$

Show also that the curve has, at right angles to the axis of x , two double tangents, the equation whereof is any one of the three equivalent forms

$$(a+d-2x)(b+c-2x) : (b+d-2x)(c+a-2x) : (c+d-2x)(a+b-2x) \\ = \rho^2 : \sigma^2 : \tau^2 \dots\dots\dots 69$$

4417. (By S. Roberts, M.A.)—Two conics have a common focus, about which one is turned; show that the envelope of the common chords is a conic having a focus at the given focus 117

4722. (By Professor Cayley, F.R.S.)—1. Show that the conditions in regard to the reality of the roots of the equations $(x^2 - a)^2 + 16A(x - m) = 0$, are, if

$$(4m^2 - 3a)^3 - (8m^3 - 9ma - 27A)^2 = -, \\ \quad \quad \quad b$$

then the roots are two real, two imaginary; but if

$$(4m^2 - 3a)^2 - (8m^3 - 9ma - 27A)^2 = +,$$

then, if simultaneously $a = +$, $A (ma - 9A) = +$,

the roots are all real, but otherwise they are all imaginary.

2. If the roots of the foregoing equation are all imaginary, then for any real value whatever of y , the roots will be all imaginary of the equation

$$(x^2 + y^2 - a)^2 + 16A(x - m) = 0 \quad 49$$

4667. (By the late Professor Clifford, F.R.S.)—If a conic be inverted into a circular cubic with a double point, prove that the foci and directrices of the conic will invert into foci and directing circles of the cubic
..... 33

4862. (By Professor Martin, M.A.)—On a horizontal plane there is a hill in the form of a spherical segment, whose base is $2a$ rods in diameter, and its altitude or perpendicular height is b rods. A roadway winds round this hill, so as to be inclined at every point at angle β to the horizontal plane. Find (1) the curve the road is, (2) the whole length of the road, and (3) how high up the hill the road can ascend..... 44

4897. (By the late Professor Clifford, F.R.S.)—Let $U, V = 0$ be the point-equations, and $u, v = 0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u \pm \lambda v = 0$, they will intersect on $U - \lambda^2 V = 0$. This last conic passes through the points of contact of the conics $u \pm \lambda v = 0$ with the common tangents of u and v 71

4971. (By Professor Sylvester, F.R.S.)—The equation to the inverse of a conic in Cartesian coordinates being given, find the algebraical expression for the node in its simplest terms..... 34

5143. (For enunciation, see Quest. 3083) 47

5250. (By the late Dr. Booth, F.R.S.)—Through a given point within an angle draw (1) the minimum line between the sides of the given angle; and (2) show that, when the angle is a right angle, the minimum line drawn through the given point is a tangent to the quadrantal hypocycloid passing through the same point..... 85

5529. (By the late T. Cotterill, M.A.)—The curve $r = a \sin \frac{1}{2}(\lambda\pi + \theta)$, where λ is an integer, being a pedal of a bi-cusped epicycloid, is unicursal; find (1) the positions of its singular points and of its four real bi-tangents; and (2) give a simple construction for tracing the curve 33

5585. (By L. H. Rosenthal, B.A.)—If we have given

$$(p^2 - 4r)(q^2 - 4t) = (pq - 2s)^2,$$

(1) solve completely the sextic equation,

$$x^6 + px^4 + qx^3 + rx^2 + sx + t = 0;$$

and give (2) the corresponding condition of solution, when the second term is *not* missing..... 94

5618. (By Professor Seitz, M.A.)—The perimeter of an inscribable quadrilateral is $2a$; show that (1) its average area is $\frac{1}{2}\pi a^2 (\frac{1}{16}\pi - \frac{1}{2}\pi)$; (2) if the quadrilateral is also circumscribable, its average area is $\frac{1}{8}\pi a^2$; and (3) the average area of the inscribed circle is $\frac{1}{32}\pi a^2$ 112

5650. (By Professor Malet, F.R.S.)—Prove (1) that the locus of the poles of the sides of the triangle of reference with respect to a variable

conic which circumscribes it, and which also touches the fixed conic $ax^2 + by^2 + cz^2 = 0$, is the curve $ax^2 + by^2 + cz^2 = 0$. Hence (2) show that we can inscribe in this curve a triangle whose sides pass through the vertices of the triangle of reference, any point on the curve being assumed as one vertex. Prove also (3) that, if from any point on one side of the triangle of reference we draw tangents to this curve, the line joining their points of contact passes through the opposite vertex 79

5689. (By Professor Cayley, F.R.S.)—Show (1) that the apparent contour of a Steiner's surface ($2xyz + y^2z^2 + x^2z^2 + x^2y^2 = 0$), as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $(\pm 1, 0)$ and $(0, \pm 1)$; and (2) find the surface-contour, or curve of contact of the cone and surface 81

5712. (By Professor Malet, F.R.S.)—Express in terms of Elliptic Functions the complete integral of the linear equation

$$\frac{d^2y}{dx^2} + \frac{2(1-2x)}{x(1-x)} \frac{dy}{dx} + \frac{1-7x+7x^2}{x^2(1-x)^2} \frac{dy}{dx} - \frac{1-2x}{2x^2(1-x)^2} y = 0 \dots\dots 80$$

5820. (By Professor Sylvester, F.R.S.)—If $x^3 + y^3 + z^3 = 3xyz$, and $x^6 + y^6 + z^6 - x^2y^2 - x^2z^2 - y^2z^2 = 0$; show that (1) the variables will bear 18 distinct ratios to each other, if their order be taken into account; and (2) if their order be disregarded, these are reducible to the three following ratios, where we have:—

$$q = \left(\frac{27}{\rho^2} - 1\right)^{\frac{1}{2}} \text{ and } \rho^2 = 1.$$

$$(1+q)^{\frac{1}{2}} : (1+qp)^{\frac{1}{2}} : (1+q^2p)^{\frac{1}{2}}, \quad (1+q)^{\frac{1}{2}} : (1+qp)^{\frac{1}{2}} \rho : (1+q^2p)^{\frac{1}{2}} \rho^2, \\ (1+q)^{\frac{1}{2}} : (1+qp)^{\frac{1}{2}} \rho^2 : (1+q^2p)^{\frac{1}{2}} \rho,$$

[This Question contains virtually the complete analytical solution of the problem of finding the points in which a given cubic is osculable by other cubics in 9 consecutive points.] 109

6197. (By Professor Morel.)—Résoudre $x^6 - x^5 - 7x^3 + x^2 + 6 = 0$. 112

6221. (By Professor Wolstenholme, M.A.)—Trace the curve

$$\frac{12c}{\pi} \int_0^{\pi} \frac{\sin z \sin 2z \sin 3z \sin 4z}{z^4} \cos \frac{\pi x}{a} dz,$$

and prove that its area (between the curve and the axis of x) is $288c^2$. 53

6343. (By Professor Tanner, M.A.)—Solve the equations

$$2u_n v_n = 2u_{n-1}^2 = (u_n + u_{n-1}) v_{n-1} \dots\dots\dots 111$$

6542 & 6628. (By Professor Seitz, M.A.)—A plane is passed through three random points within a sphere; show (1) that the average area of the section is $\frac{1}{2}\pi r^2$, and (2) that the average volume of the smaller of the two segments into which the sphere is divided is $\frac{1}{16}\pi r^3$ 43

6567. (By Professor Nash, M.A.)—In Charles's *Aperçu Historique*, p. 342, there is given the following method of generating a Cartesian oval, or "ligne aplanétique":—

"Etant donné un cercle et un point fixe, pris arbitrairement dans son plan; si par ce point on mène un rayon vecteur à un point quelconque de la circonférence du cercle, et une seconde droite, qui fasse avec un certain axe fixe un angle double de celui que fait le rayon vecteur avec cet axe,

et qu'on porte sur cette seconde droite, à partir du point fixe, un segment proportionnel au carré du rayon vecteur; l'extrémité de ce segment aura, pour lieu géométrique, une ligne aplanétique formée de deux ovales conjugués, dont un foyer est au point fixe."

Prove that this method of generation gives only Limaçons, that is, Cartesians in which two single foci coalesce. The fixed point is the single focus of the Limaçon, and the line joining the fixed point to the centre of the circle bisects the angle between the given line and the axis of the Limaçon. The node will be a crunode if the given point lie within the given circle, and an acnode if it lie without the circle. If the point lie in the circumference of the circle, the node becomes a cusp, and the curve a cardioid

6600. (By Professor Seitz, M.A.)—A plane triangle is formed by joining three points taken at random in a given semi-circumference;

show that the average area (1) of the triangle is $\frac{3r^2}{\pi} \left(1 - \frac{8}{\pi^2}\right)$, and (2) of its inscribed circle is $\pi r^2 \left(\frac{5}{2} - \frac{162}{\pi^2} + \frac{432}{\pi^3}\right)$

6691. (By Professor Hudson, M.A.)—Prove (1) that, if the curve $r = a + b \cos \theta$ have a point of inflexion, a must lie between b and $2b$; and (2) trace the change in form, as a changes from 0 to $2b$

6648. (For enunciation, see Quest. 6542)

6731. (By W. J. C. Sharp, M.A.)—If any number of particles attract each other with a force varying directly as their masses, and inversely as the cube of the distances between them; prove that, whatever the origin and axes may be $\sum m(x^2 + y^2 + z^2) = at^2 + bt + c$

6756. (By E. B. Elliott, M.A.)—If u be a homogeneous function of the n^{th} degree in x_1, x_2, x_3, x_4 , four variables u_1, u_2, u_3, u_4 its first, and $u_{11}, u_{12}, \dots, u_{44}$ its second partial differential coefficients, prove (1) that

$$\begin{vmatrix} u_{11} & \dots & u_{14} \\ \dots & \dots & \dots \\ u_{41} & \dots & u_{44} \end{vmatrix} = \frac{1}{x_4^2} \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_1 \\ u_{21} & u_{22} & u_{23} & u_2 \\ u_{31} & u_{32} & u_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} + \frac{n(n-1)}{x_4^2} u \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix};$$

and (2) corresponding relations hold for any number of variables

6764. (By the Rev. T. R. Terry, M.A., F.R.A.S.)—Show that, unless m is a negative integer, the limit, when p is indefinitely increased, of

$$\frac{m(m+1)(m+2) \dots (m+2p-1)}{1 \cdot 3 \cdot 5 \dots (2p-1) 2m(2m+2) \dots (2m+2p-2)} \text{ is } 2^{m-1} \dots$$

6768. (By Professor Crofton, F.R.S.)—Prove that

$$f(x+hD) \cdot 1 \equiv f(x) + \frac{h}{2} f''(x) + \frac{h^2}{2 \cdot 4} f'''(x) + \&c. \dots$$

6790. (By G. F. Walker, M.A.)—Three particles are at the corners of an equilateral triangle of side a , and are each projected parallel to the opposite side towards the same parts with a velocity $[af(a)]^{\frac{1}{2}}$, where $f(r)$ is the law of attraction between the particles; show that they will always be at the corners of an equilateral triangle

6809. (By J. L. McKenzie, B.A.)—(1) Let O be the orthocentre of a triangle ABC , and N the centre of the nine-point circle. Let the circle through O , and through the middle points of OB and OC , cut the line NO in P ; and let the two other analogous circles cut NO in Q and R . Prove that the lines joining P , Q , and R with the middle points of BC , BA , and AB , all intersect in a point S on the circumference of the nine-points circle. (2) Let the circle through N , and through the middle points of OB and OC , cut NO in P' , and similarly for two other circles; prove that the lines joining P' , Q' , R' to the feet of the perpendiculars on BC , CA , and AB from the opposite angles, all intersect in the same point S as before

..... 39

6817. (By T. C. Simmons, M.A.)—From Question 6720 deduce, by reciprocation, a property of the parabola, and give an independent proof of the theorem thus obtained

..... 64

6856. (By Professor Townsend, M.A.)—In the irrotational strain of an incompressible lamina in the plane of its mass, if the equations $r^n \cos n\theta = a^n$ and $r^n \sin n\theta = b^n$, where a and b are parameters, represent the line-systems of greatest and least displacement of the molecules; show that the equations $r^{4n} \cos \frac{1}{2}n\theta = a^{4n}$ and $r^{4n} \sin \frac{1}{2}n\theta = b^{4n}$, where a and b are parameters, represent the line-systems of greatest and least dilatation of the strain

..... 50

6864. (By Professor Hudson, M.A.)—If I , O be the centres of the inscribed and circumscribed circles of a spherical triangle r , R their radii, and $2s = a + b + c$, prove that $2 \sin s \cos IO = (\sin a + \sin b + \sin c) \cos r \cos R$

..... 58

6865. (By G. F. Walker, M.A.)—Two equal particles are connected by a string which passes over a smooth peg; prove that, if one move vertically, and θ be measured from the vertical downwards, and u be the reciprocal of the radius vector, the equation of motion of the other is

$$\frac{\sin \theta}{u^3} + \frac{d}{d\theta} \left\{ \frac{\sin^2 \frac{1}{2}\theta + \frac{\sin \theta}{u} \frac{du}{d\theta}}{u^2 \left(2 \frac{d^2 u}{d\theta^2} + u \right)} \right\} = 0 \quad \dots\dots\dots 65$$

6872. (By R. Knowles, B.A., L.C.P.)—In a central conic, if GK is the perpendicular from G , the foot of the normal at P on CP drawn from the centre C , prove that the rectangle $CP \cdot PK$ is constant

..... 38

6876. (By the Rev. W. A. Whitworth, M.A.)—Without altering the perimeter, it is required to lengthen very slightly one of the sides of an equilateral triangle and to shorten another, so that the radius of the inscribed circle may become commensurable with the sides. The increment and decrement, in order to be inappreciable to the eye, must not exceed $\frac{1}{177}$ th part of the length of the sides. Determine the ratio in which the sides may be altered

..... 43

6878. (By T. C. Simmons, M.A.)—If A , B are two points in the plane of a given circle; give a geometrical construction for finding the position of a point P on the circumference, such that the lines AP , BP make equal angles with the tangent at P

..... 58

6879. (By J. O'Regan.)—The horizontal base AB of a triangle being given in magnitude and position, and the side AC in magnitude only,

construct the triangle so that a body may fall along CB from C to B in the least time possible 63

6883. (By T. Morley, L.C.P.)—If the water-section of a ship be taken for the base of a right cylinder equal to the volume of the ship, and H be the height of the cylinder; prove that the vertical oscillations of the vessel are the same as the vibrations of a simple pendulum whose length is SH; S being the specific gravity of ship and cargo 68

6894. (By Professor Seitz, M.A.)—Three points are taken at random within a circle; prove that the mean value of the radius of the inscribed circle of the triangle formed by joining them is $\frac{192r}{35\pi^2} \left(\frac{128}{135} - \frac{\pi^2}{12} \right)$... 66

6895. (By Professor Matz, M.A.)—Find the value of ϕ in the equation $\pi \tan^2 \phi + \sqrt{(2) - \sqrt{(3)}} = -3 [\pi \sqrt{(6)} \tan^2 \phi]^{\frac{1}{3}}$ 40

6917. (By J. L. McKenzie, B.A.)—A straight line XY cuts the tangents at two given points A and B on a given circle in L and M respectively, and the circle itself in C and D; then (1) if the circles ABL, ABM cut XY in α and β respectively, prove $Ca = D\beta$; and (2) if the circles passing through L and M, and touching the given circle in B and A respectively, cut XY in α' and β' , prove $Ca' = D\beta'$ 61

6924. (By Professor Seitz, M.A.)—A triangle is formed by joining three random points within a sphere; prove that the average area (1) of the triangle is $\frac{1}{4}\pi r^2$, and (2) of its inscribed circle is $(\frac{3}{2}\frac{\pi}{3}\frac{1}{3} - \frac{1}{2}\frac{\pi}{3} \log 2) \pi r^2$... 60

6932. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—It is an old theorem that every $(12+r)$ -edron P, whose summits are all triedral, and whose faces are all 5-gons and 6-gons, has twelve pentagons. Required the number of these solids (P) for values of r up to $r=7$, no one being counted which can be drawn in any position either as the repetition or the reflected image of another; together with a complete account of symmetry, enumerating the symmetrical faces, summits, and edges. The required number is less than 20 35

6933. (By J. J. Walker, M.A.)—A homogeneous rectangular parallelepiped, the edges of which are a, b, c , floats in a liquid (the ratio of the densities of the two being $\rho : 1$) with the edge a vertical; show that, if displaced through an angle θ (the top remaining above the surface of the liquid), so that the plane of ab remains parallel to itself, the body will right itself if $a^2 + b^2$ is less than $(1 + \sec^2 \theta) \div 12\rho(1 - \rho)$; and hence find the condition of stability for small displacements 54

6944. (By W. Gallatly, B.A.)—Prove that, in any triangle ABC,

$$\frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} + \frac{\sin \frac{1}{2}(C-A)}{\sin \frac{1}{2}(C+A)} + \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

$$+ \frac{\sin \frac{1}{2}(B-C) \cdot \sin \frac{1}{2}(C-A) \cdot \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(B+C) \cdot \sin \frac{1}{2}(C+A) \cdot \sin \frac{1}{2}(A+B)} = 0$$
 41

6947. (By G. F. Merritt, M.A.)—The tangent and normal to an ellipse at the point P cut the axes in T and t, G and g, respectively; GT, gt are bisected in O and o; Oo cuts the normal Pg in Q: show that PQ is one-half of the radius of curvature at P 120

6956. (By Professor Malet, M.A.)—If A, B, C, H, &c. be the minors of the discriminant of the quadric of which the quadriplanar equation is

$$aa^2 + b\beta^2 + c\gamma^2 + d\delta^2 + 2ha\beta + 2ga\gamma + 2f\beta\gamma + 2la\delta + 2m\beta\delta + 2n\gamma\delta = 0,$$

ch taken with respect to the corresponding small letter, prove that the equations (1) of the quadric, (2) of the tangent line through $(\alpha, \beta, \gamma, \delta)$, (3) of the pair of tangent planes through the line joining the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$, are given by equating to zero the respective determinants

$$\begin{vmatrix} A & H & G & L & \alpha \\ H & B & F & M & \beta \\ G & F & C & N & \gamma \\ L & M & N & D & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}, \quad \begin{vmatrix} A & H & G & L & \alpha' & \alpha \\ H & B & F & M & \beta' & \beta \\ G & F & C & N & \gamma' & \gamma \\ L & M & N & D & \delta' & \delta \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & 0 \end{vmatrix},$$

$$\begin{vmatrix} A & H & G & L & \alpha' & \alpha'' & \alpha \\ H & B & F & M & \beta' & \beta'' & \beta \\ G & F & C & N & \gamma' & \gamma'' & \gamma \\ L & M & N & D & \delta' & \delta'' & \delta \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 & 0 \\ \alpha'' & \beta'' & \gamma'' & \delta'' & 0 & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & 0 & 0 \end{vmatrix} \dots\dots\dots 103$$

6964. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

When the Sultan of Borneo holds a review
Upon New Year's Day, the best of the splendour
Resides in a smoking cap, jewelled and new,
Which by olden law the Vizier must render.

The gift is a fez having twenty-four faces,
Counting the pentagon girding the head royal,
Silks in a rainbow: and gorgeous laces,
With diamonds ablaze on the lines polyedrial.

Forty-two provinces quake at his name;
Forty-two scutcheons blazon their fame.
Of shields pentagonal glitter a score,
Of heptagonal six, octagonal four,
And of heralding hexagons one dozen more.

In the twenty-four shields may be any of these,
With blazoning braudric as you please,
Aslant or askew,
If headed true

They star all the angles in forty-four threes.

Say now, what is royalty more than a joke,
If it cannot be lord of the fashion in smoke?

All smoke, to be loyal and orthodox,
Must kiss a right rig of the knowledge-box,

With braiding after the King's:
(With head-gear amiss, and those flaws in your frocks,
Your smoke and your censers
Are bungled pretence, sirs)

Who copies a cap not a three-year-old—swings;
The old, on big dummies, fill two palace wings.

Poor Vizier! he finds it a burden sore,
To con those numberless dummy-caps o'er,
In mortal fear and dread;

For thus the statute of caps is read:—
If any two, placed on their pentagon brimming,
And matched apart from the pranking and trimming
Have shields in angles and order of limning
Alike o'er the base, off goes his head!

The last review is a horrible story :—
 As side by side on a golden stand
 Were mirror and cap, and the king was about
 To unloose the storm of his thundering band
 By donning that dazzle of glory—
 Treason! they shout,
 And the foe rushes out,
 With a battered old fez in his hand.
 As slow in the mirror its image turns,
 Lo! lo! into fierce and fiercer flame
 In the Sultan's sight the evidence burns,
 That the braided shields in order and name
 On the new cap and image are all the same.
 Up rose the Sultan with murderous eye,—
 (I saw the Vizier begin to die).—
 "O insult and wrong for a Monarch to bear!
 Was ever a monster of treason so mad? O
 Grandfather of asses, who bidd'st me wear
 Not even a second-hand hat, but its shadow!"
 Then, as trumpets and drums and yells make the coast ring,
 Down goes the Vizier agape in the bow-string.
 What a scandalous waste both of heads and of caps!
 Mere counting of dummies might end it, perhaps,
 If the tyrant could somehow politely be told
 The possible number of caps, new and old.
 Then find it and send it him; best, in propriety,
 Signed by the Secs. of the Royal Society.

..... 118

6966. (By the Rev. W. A. Whitworth, M.A.)—Show that there are 220 triangles whose sides are integers not exceeding 100, and whose areas are also integral..... 37

6967. (By T. Woodcock, B.A.)—A ray of light passes through a medium in which the refractive power at any point is a single-valued function of the distance from a fixed point; show that all the normals that can be drawn from the fixed point to the path of the ray are situated at equal angular intervals, and have one of two lengths alternately... 123

6971. (By G. S. Carr, M.A.)—If a number of voltaic cells differing both in electro-motive force and conductivity be joined "in multiple arc," show that the difference of potentials of the electrodes of the battery, when connected by a wire, is equal to the sum of the current strengths of the separate cells divided by the sum of the conductivities of all parts of the circuit 110

6976. (By J. Young, B.A.)—Draw a common tangent to a parabola and any circle intersecting it and passing through the focus of the parabola 45

6979. (By Professor Townsend, F.R.S.)—A spiral intersecting at any constant angle all the meridians of any convex surface of revolution, being supposed the frictionless catenary of a uniform cord, or the frictionless trajectory of a material particle, constrained to rest or move on the surface, under the action of a force passing perpendicularly in every position through the axis of revolution; show that the force varies, for the catenary inversely as the square, and for the trajectory inversely as the cube, of the distance from the axis 31

6989. (By R. Tucker, M.A.)—Show that, in the curve whose equation

is $b^2e^{\frac{y}{b}} + a^2e^{-\frac{y}{b}} = 2a^2 \cos \frac{x}{b}$, there are pairs of radii of curvature (ρ, ρ') , so related that $\rho^2 + \rho'^2 = a^2 + b^2$ 40

6993. (By G. Heppel, M.A.)—A conic passes through A, B, C. The tangents at those points intersect in O, P, Q. They intersect the opposite chords in T, U, V, which are in a straight line. Show that

$$\pm \frac{1}{AUV} \pm \frac{1}{BVT} \pm \frac{1}{CTU} + \frac{4}{OPQ} = \frac{1}{ABC},$$

the upper sign being taken where the lines AB, AC, &c. are produced both forwards or both backwards to meet the line TUV, the lower sign where one is produced forwards and the other backwards 123

6994. (By W. Nicholls, B.A.)—A rigid uniform beam, resting on a rough horizontal plane and against an equally rough vertical wall, is prevented from slipping by a string attached to the beam and to the intersection of the planes; the beam lying in a vertical plane perpendicular to the wall, to find the tension and the limiting position of the string ... 81

7001. (By T. Woodcock, B.A.)—The straight line through P, any point on the circumscribing circle of a triangle ABC, drawn perpendicular to the pedal line of P, meets the sides of the triangle in x, y, z : and the straight line through the orthocentre, parallel to the pedal line of P, meets the sides in X, Y, Z; Pp, Pq, Pr are drawn at right angles to PX, PY, PZ to meet the corresponding sides in p, q, r ; show that (1) xp, yq, zr are bisected by the pedal line of P; (2) the triangle pqr is double of the triangle ABC; (3) PAq, PrA; PBr, PpB; and PCp, PqC are pairs of similar triangles 39

7033. (By Professor Hudson, M.A.)—A particle is attracted to one and repelled from another centre of force varying inversely as the square of the distance and of equal absolute intensity; show (1) that the surfaces of equilibrium are those formed by the revolution of parts of the curves

$$(r^2 + a^2)^3 (r^2 + a^2 - 2c^2) + 4a^2r^2 \cos^2 \theta [c^4 - 2(r^2 + a^2)(r^2 + a^2 - c^2)],$$

$$+ 16a^4r^4 \cos^4 \theta = 0,$$

where $2a$ is the distance between the centres of force; and find (2) what the other parts of these curves represent. ... 32

7034. (By the Editor.)—If ACB be a circle on AB as diameter, and P, P' two points on the vector AC such that CP = CP' = distance (CD) of C from AB; show that (1) the locus of P is a heart-shaped curve circumscribing the circle, and of P' a continuous loop inscribed therein;

(2) the areas of these loci, in parts of the area of the circle, are $\frac{5}{4} \pm \frac{8}{3\pi}$;

(3) the maximum vector AP₁ ($= \frac{2}{3}\sqrt{3} \cdot AB$) makes an angle P₁AB = 30° with AB; (4) the maximum tangent P₂T from P to the circle makes an angle P₂AB = 39° 49' 15" with AB; and (5) the revolution of AP₂ round AB generates the cone of greatest entire surface that can be cut from the sphere on AB as diameter..... 86

7037. (By the Editor.)—A particle moves on the surface of an ellipsoid under the action of an attractive force varying inversely as the cube of the distance from the plane containing the greatest and least axis; show that, by a proper arrangement of the initial cir-

cumstances, the path will be one of the circular sections of ellipsoid passing through the centre 69

7047. (By R. Russell, B.A.)—Show (1) generally how to solve

$$(a_1 b_1 c_1 f_1 g_1 h_1)(x, y, z)^2 = l_1, \quad (a_2 b_2 c_2 f_2 g_2 h_2)(x, y, z)^2 = l_2, \\ (a_3 b_3 c_3 f_3 g_3 h_3)(x, y, z)^2 = l_3;$$

and (2) prove that the solution of

$$x^2 + yz = a, \quad y^2 + zx = b, \quad z^2 + xy = c,$$

depends on the solution of $u^2 + v^2 + w^2 = 5uvw$, where

$$u = (b-c) \left(1 + \frac{\lambda}{a}\right), \quad v = (c-a) \left(1 + \frac{\lambda}{b}\right), \quad w = (a-b) \left(1 + \frac{\lambda}{c}\right).$$

[This is connected with the Editor's Questions 6928 and 6972.] ... 25

7060. (By R. Tucker, M.A.)—Three parabolas are described about the triangle ABC, so that A is the vertex of the diameter bisecting BC, and so on; show that (1) if p, q, r are the respective parameters of the diameters, then $\frac{1}{2}[p^{-2} + q^{-2} + r^{-2}] = 2\sum a^2 \cdot \sum a^{-4} - 3\sum a^{-2}$(a),

$$a^4 p^{-2} + \dots = 12(a^2 + b^2 + c^2), \quad a^2 p^{-4} + \dots = 144(a^4 + b^4 + c^4).....(\beta, \gamma);$$

(2) when ABC is an equilateral triangle, the area of the triangle formed by joining the foci = $\frac{3}{4}\Delta ABC$ 53

7066. (By Dr. Macfarlane, F.R.S.E.)—Show that the number of ways in which an elector may vote, under the cumulative system of voting, when there are p candidates for q seats, is $\frac{(p+q)!}{p! q!}$. [Applied to the recent election of a School Board for Dundee, this result shows that the number of possible ways of voting was 43 times the population of the globe.] ... 25

7071. (By T. C. Simmons, M.A.)—Prove the two following constructions for finding the vertical chord of curvature at any point P of a parabola; (1) on PA produced take AR equal to three times the projection of the abscissa of P on AP; (2) from the focal distance SP cut off SV = AS, and on PS produced take SW = 3PS: draw a circle through the points A, V, W, meeting PA produced in R. In each of these cases, PR will be the required chord of curvature 67

7074. (By Professor Mannheim.)—On a n points fixes $f, f', f'', \dots$, et le courbe (m) , lieu des points tels que m , pour lequel on a $mf \times mf' \times mf'' \dots = \text{const.}$ Soient μ le centre de courbure de (m) pour le point m ; p le point de rencontre de la normale en m à (m) et de la perpendiculaire élevée de f à fm ; $p', p'', \dots$ des points analogues à p pour $f, f', f'', \dots$

Démontrer que $\frac{1}{m\mu} \geq \left(\frac{1}{mp}\right) = 2\sum \left(\frac{1}{mp^2}\right) - \sum \left(\frac{1}{mf^2}\right)$ 21

7075. (By Professor Wolstenholme, M.A.)—If in a triangle ABC there be inscribed another triangle abc , and two conics through Bca, Cab intersect again in A', B', C' ; prove that the conic inscribed in ABC and touching two sides (A'B', A'C') of the triangle A'B'C' will have double contact with the conic through the five points a, b, c, B', C' 34

7081. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

"Apollo tun'd the lyre; the Muses round
With voice alternate aid the silver
sound,"

Yes—that sufficed of yore; but ah!
Since roaming demigods have seen the
ballet
And heard amazed the quiring storms of
Halle,
Apollo swears, and Pallas vows,
That heaven must work and try to
rouse
Their ever drowsier old Papa
From sleeping like a log.
So all their godships are agog
On dancing and sol-fa.

The Muses and the Charites,
By Phoebus lessoned long,
In all the lore of dance and song
Of course completely are at ease.
Not so the Heroes; many a moon
Of practising from dawn till noon
Must wax and wane, before they shine
For all the teaching of the Nine,
In either grace or tune.

The rage in heaven, of late begun,
For ballet and concert all in one,
May grow, but, as yet,
It is all quartet,
Or rather a double duet.

When Jove in his cups begins to doze,
To one couple dancing, one couplesings,
While Momus and Orpheus rouse the
strings,
Ten minutes or so; and 'tis o'er
With a chorus of all at the close,
As on come another four:
And so on for hours and hours it goes.

Nor dance nor sing together again,
That night, can any coupled twain.

The Graces and Muses are all in the bill,
With gods or demigods fewer or more,
Of passable shape and skill,
Selected for training at Erato's will,
From two to many a score.

Five times each goddess divinely will sing,
Five times more enchantingly dance;
Six times each demigod's voice shall ring,
Six times that hero must prance;
All each time o'er in a different four.

The gods in a four may be all, none, or any;
(On quartets of heroes,
O spare me your *quæro's*!)

You may band any number with god-
desses twelve:
But at the beginning, 'tis Pallas's plea-
sure,
Contriving what failure she may to
shelve,
(The order in fact is
Conducive to practice,
And pushing their godships in tune and
in measure.)

That two are too many,
But one may be there,
To sing to each dance of a goddess-pair.

This rule makes the riddle: this keeping
in view,
Write out the programmes, say a dozen,
enow
For two to twelve Heroes; then nothing
is easier,
After its kind, than the task of the Vizier

..... 118

7083. (By Dr. Macfarlane, F.R.S.E.)—Prove that

$$\frac{1}{[a^2 + b^2 + 2ab \cos(\alpha - \beta)]^n} \cos n \tan^{-1} \frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta} = \frac{1}{a^n} \left\{ \cos n\alpha \right. \\ \left. - n \frac{b}{a} \cos [(n+1)\alpha - \beta] + \frac{n(n+1)}{2!} \frac{b^2}{a^2} \cos [(n+2)\alpha - 2\beta] - \&c. \right\} \\ = \frac{1}{(a \cos \alpha + b \cos \beta)^n} \left\{ 1 - \frac{n(n+1)}{2!} \left(\frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta} \right)^2 \right. \\ \left. + \frac{n(n+1)(n+2)(n+3)}{4!} \left(\frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta} \right)^4 - \&c. \right\}. \quad 78$$

7084. (By J. Griffiths, M.A.)—Find the locus of intersection of three mutually perpendicular planes to an elliptic paraboloid 26

7087. (By G. S. Carr, M.A.)—If a number of voltaic elements, all differing in electro-motive force and resistance, be arranged in unequal groups, the cells of each group in multiple arc and the groups in a series; show that the current which passes through a wire connecting the poles of the battery will be given by the following rule. Multiply the sum of the separate current strengths of the cells in each group by the effective resistance of the group, and divide the sum of all the products so formed by the total effective resistance of the circuit 110

7091. (By W. J. C. Sharp, M.A.)—Show that (1) if the decimal equivalent to a fraction whose denominator is n have the greatest possible number of figures ($n-1$) in the period, this is the same for all such fractions; (2) when this is the case, n is a prime number; and (3) no fraction with a denominator n can have $n-2$ figures in the period of the corresponding decimal 108

7099. (By the Editor.)—Sum (1) the series

$$\frac{\sin^2 \theta}{1^4} - \frac{\sin^2 3\theta}{3^4} + \frac{\sin^2 5\theta}{5^4} - \&c.$$

between the limits $\theta = 0, \theta = 2\pi$; and $\theta = \frac{1}{2}\pi, \theta = \frac{3}{2}\pi$; and hence

show (2) that $\pi^4 = \frac{3 \cdot 2^{\frac{1}{2}}}{11} \left(\frac{1}{1^4} - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \&c. \right)$ 23

7101. (By Professor Townsend, F.R.S.)—A cubic surface being supposed to have a nodal point; show that, of the three triple tangent planes intersecting in any one of its twenty-seven superficial lines, the three points of contact not on the line connect with the nodal point by a triad of conjugate diameters of the nodal cone 23

7112. (By R. F. Scott, M.A.)—Three circles, which pass through the same point O, intersect again in A, B, C. Through the point A a line is drawn meeting the circles which intersect at A in D and E. The straight lines DB, EC meet the third circle again in F and G. Prove that, as the line DAE varies, the straight line FG envelopes a circle concentric with the circle BOC 29

7115. (By A. McMurchy, B.A.)—If $a_1, a_2, a_3 \dots$ be the lengths of the sides of a convex polygon ABCD ... inscribed in a circle; $p_1, p_2, p_3 \dots$ the lengths of the perpendiculars on the considered position from any point P on the smaller arc cut off by a_1 ; prove that

$$\frac{a_1}{p_1} = \frac{a_2}{p_2} + \frac{a_3}{p_3} + \dots + \frac{a_n}{p_n} \dots\dots\dots 30$$

7120. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

With the last [*i.e.*, Quest. 7081], should the riddle give any delight,
I can offer another, if Phoebus be right.

Read—Two are too many gods—one must be there,
To sing at each dance of a goddess-pair.

Terpsichore deems this cannot be done;

Minerva maintains that it may,
And never a Muse or a Grace away,
With squads heroical more than one,
Not numbering more than they.

One more, and this one cannot puzzle you long;

Urania moves it; Apollo decrees;

(Then oh! what a shout from the demigods rose!

It frightened Orion and shook the sky,

From Scorpio to Gemini):

If all the gods in dance and song—

The rules retained

Already explained—

Shall so have improved at the season's close,

As the beautiful twelve to please,

That every four that glorious night

Shall be heroes three and a goddess bright.

..... 118

7122. (By Professor Townsend, F.R.S.)—A point, taken arbitrarily

in the plane of a tetrad of circles, every two of which intersect at right angles, and three of which are consequently real while the fourth is imaginary, being supposed to undergo quadruple inversion with respect to the four circles of the tetrad taken in any order; and the several inversions being supposed performed for the twenty-four different orders in which the circles may be taken in succession; show that

(a) For every separate order in which the circles may be taken, the point returns after the fourth inversion to its original position in the plane.

(b) For the twenty-four different orders, the point and its several inverses form a group of eight points divisible into pairs of concyclic tetrads in six different ways 116

7125. (By Professor Wolstenholme, M.A.)—Prove that, if

$$y = \frac{1}{\epsilon^x - 1}, \text{ then } \frac{d^2 y}{dx^2} = (-1)^n \left\{ \frac{A_1}{\epsilon^x - 1} + \frac{A_2}{(\epsilon^x - 1)^2} + \dots + \frac{A_{n+1}}{(\epsilon^x - 1)^{n+1}} \right\},$$

where $A_p = \Delta^p 0^n + \Delta^{p-1} 0^{n-1} \dots \dots \dots 26$

7166. (By Professor Townsend, F.R.S.)—The central axis of a system of three forces in a common space being supposed to pass through the centre of the quadric determined by their three lines of action in the space; show, on elementary principles, that it must coincide with a principle axis of the surface 85

7167. (By Professor Minchin, M.A.)—A rigid body has uniplanar motion; find the locus, at any instant, of points the directions of whose velocities make a given angle with the directions of their accelerations. 22

7171. (By E. W. Symons, M.A.)—Prove that, in any triangle,
 $\cot^2 A + \cot^2 B + \cot^2 C > 1 \dots \dots \dots 29$

7175. (By G. G. Morrice, B.A.)—Prove that the angle between any two straight lines in space $= \sin^{-1} |l_1 m_2 n_3|$, where $l_1 m_1 n_1$, $l_2 m_2 n_2$ are their direction-cosines, and $l_3 m_3 n_3$ those of their shortest distance 28

7179. (By the Rev. G. Richardson, M.A.)—If AOD be the perpendicular from A on the opposite side of the triangle ABC, O being the orthocentre, I the centre of the inscribed circle, r its radius; prove that $IO^2 = 2r^2 - AO \cdot OD \dots \dots \dots 27, 99$

7181. (By H. L. Orchard, M.A.)—Given that the sum of the radii of the inscribed, escribed, and circumscribed circles of a certain equilateral triangle is unity, find the area of the triangle 28

7185. (By Morgan Jenkins, M.A.)—Show, as a special case of Gergonne's general construction for describing a circle to touch three given circles (Salmon's *Conics*, Ch. ix., Art. 127), that the nine-points-circle of a triangle touches the three escribed circles of that triangle; also, in like manner, that it touches two of the escribed circles and the inscribed circle; show also that the three circles, each of which touches two of the escribed circles externally and one internally, pass through the same point, viz., the centre of the circle inscribed in the triangle obtained by joining the middle points of the sides of the first triangle 88

7186. (By R. Tucker, M.A.)—If A' , B' , C' , (D , E , F) are the mid-points respectively of the arcs (and sides) BC , CA , AB of a circle (and triangle) ABC ; prove that (1) the pedal lines of A' , B' , C' with reference to the triangle ABC form a triangle $A''B''C''$ whose sides are equal and parallel to the sides of $A'B'C'$, and $A''D$, $B''E$, $C''F$ meet in the ortho-

centre of $A'B''C''$; and (2) if P, P' are any two diametral points, their four pedal lines with reference to the triangles $ABC, A'B'C'$ form a rectangle..... 121

7188. (By Dr. Curtis.)—Prove that the centre of gravity of the area included by a loop of the curve $r = a \cos n\theta$ is at a distance from the origin

$$= 16an^4 \sin \left(\frac{n+1}{2n} \pi \right) + \{ (n^2-1) [(3n)^2-1] \pi \}.$$

[The cases for $n = 2$ and $n = 3$ are exercises in Todhunter's *Statics*.] 27

7197. (By the Editor.)—With two given sides, construct a triangle such that the square of the sine of half the angle contained by the sides may be equal to the squares of the sines of the other two semi-angles. 56

7200. (By G. S. Carr, B.A.)—If two non-intersecting circles, having radii a, b , centres A, B , and distance between the centres c , be inverted into concentric circles having their common centre at a distance d from A in the direction AB ; show that a, b , the radii of the images, must have the values

$$\alpha = \frac{(2cd - a^2 + b^2 - c^2 + S)a}{a^2 - b^2 + c^2 - S}; \quad \beta = \pm \frac{(2cd - a^2 + b^2 - c^2 + S)b}{a^2 - b^2 - c^2 - S},$$

where

$$S = \sqrt{(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)},$$

and the + or - sign of β is to be read according as the smallest circle, radius b , lies within or without the other 63

7201. (By R. F. Scott, M.A.)—Prove that

$$\int_0^\pi x \log (1 - \sin^2 a \sin^2 x) dx = 2\pi^2 \log \cos \frac{1}{2}a \dots\dots\dots 54$$

7203. (By W. J. C. Sharp, M.A.)—In the late Professor Clifford's *Quest.* 6009 (*Reprint*, Vol. xxxiii., p. 59),—as to what is the form of a kite string,—assuming that the force of the wind is a function of the height, show how this function may be determined, so that the string shall assume a given form 99

7204. (By J. Hammond, M.A.)—If

$$\phi(x) = \phi \left(\frac{1}{x \sin^2 a + \cos^2 a} \right), \quad \int_0^\infty \frac{\phi(x) dx}{x^2 \tan^2 a + x - \sec^2 a} = 0, \\ \int_0^\infty \frac{\phi(x) dx}{(x-1)^2} = (1 + \sin^2 a) \int_0^1 \frac{\phi(x) dx}{(x-1)^2} \dots\dots\dots 98$$

7208. (By A. McMurphy, B.A.)—If n be the length of a normal chord to a surface $ax^2 + by^2 + cz^2 = 1$ at a point for which r is the distance from the centre, and p the perpendicular on the tangent plane, prove that

$$\frac{2}{np^3} - \frac{a+b+c}{p^2} + bc + ca + ab - abcr^2 = 0 \dots\dots\dots 55$$

7213. (By C. Morgan, B.A.)—Show that, in any triangle,

$$s \cdot \Sigma (r_a^{-1}) = \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C,$$

where (as usual) r_a is the radius of the circle escribed to the side a , and s is the semiperimeter 30

7215. (By A. H. Curtis, LL.D., D.Sc.)—Prove that

$$\begin{aligned}\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c. \dots ad\ inf. &= \frac{\pi^4}{90}, \\ \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \&c. \dots ad\ inf. &= \frac{(2^3-1)\pi^4}{2^3 \times 90} = \frac{7\pi^4}{720}, \\ \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \&c. \dots ad\ inf. &= \frac{2^4-1}{2^4} \times \frac{\pi^4}{90} = \frac{\pi^4}{96}; \\ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \&c. \dots ad\ inf. &= \frac{\pi^8}{3 \cdot 5 \cdot 7 \cdot 90} = \frac{\pi^8}{9450}, \\ \frac{1}{1^8} - \frac{1}{2^8} + \frac{1}{3^8} - \&c. \dots ad\ inf. &= \frac{2^7-1}{2^7} \times \frac{\pi^8}{9450} = \frac{127}{128} \times \frac{\pi^8}{9450}, \\ \frac{1}{1^8} + \frac{1}{3^8} + \frac{1}{5^8} + \&c. \dots ad\ inf. &= \frac{2^8-1}{2^8} \times \frac{\pi^8}{9450} = \frac{255}{256} \times \frac{\pi^8}{9450} = \frac{17\pi^8}{256 \times 630} \\ &\dots\dots\dots 59\end{aligned}$$

7217. (By T. Muir, M.A., F.R.S.E.)—Prove the identity

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_1 x_2 & x_2^2 \\ 0 & y & 0 & y_1^2 & y_1 y_2 & y_2^2 \\ 0 & 0 & z & z_1^2 & z_1 z_2 & z_2^2 \\ 0 & z & y & 2y_1 z_1 & y_1 z_2 + y_2 z_1 & 2y_2 z_2 \\ z & 0 & x & 2x_1 x_1 & x_1 x_2 + x_2 x_1 & 2x_2 x_2 \\ y & x & 0 & 2x_1 y_1 & x_1 y_2 + x_2 y_1 & 2x_2 y_2 \end{vmatrix} = |xy_1z_2|^3 \dots\dots 106$$

7218. (By Professor Genese, M.A.)—Prove that

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_2^2 & l \\ 0 & y & 0 & y_1^2 & y_2^2 & m \\ 0 & 0 & z & z_1^2 & z_2^2 & n \\ 0 & z & y & 2y_1 z_1 & 2y_2 z_2 & 2p \\ z & 0 & x & 2x_1 x_1 & 2x_2 x_2 & 2q \\ y & x & 0 & 2x_1 y_1 & 2x_2 y_2 & 2r \end{vmatrix} = -2\Delta \{ lX_1X_2 + \dots + p(Y_1Z_2 + Z_1Y_2) + \dots \}$$

where $\Delta = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$, X_1 is the minor of x_1 in Δ , &c.

..... 106

7219. (By Professor Sylvester, F.R.S.)—Prove that a series of continually increasing integers, no one of which is a sum of positive multiples of all or any of those that precede it, cannot be continued *ad infinitum* 85

7221. (By Professor Townsend, F.R.S.)—In the irrotational strain of an incompressible substance in the space of its mass, if the equipotential surfaces of the strain be a system of confocal ellipsoids in the space, determine the form of the potential as a function of the parameter of the system 102

7224. (By Professor Cochez.)—Démontrer que la fonction

$$ke^{2x} \{ [(x+y)^2 + c]^{\frac{1}{2}} - x - y \} = [(x+y)^2 + c]^{\frac{1}{2}} + x + y,$$

est telle que $x + y = y''$ 57

7225. (By Professor Cavallin, M.A.)—If a closed convex curve glides always in contact with two fixed straight lines, and rotates until returning to its original position; prove that the mean value of the arc com-

prised between the points of contact is the same for all curves that have the same perimeter..... 92

7228. (By Professor Hudson, M.A.)—From a point A on the outer of two concentric circles, tangents AP, AQ are drawn to the inner; AP, QP meet the outer again in T, R; prove that $RP : RQ = RT^2 : RA^2$. 66

7229. (By Professor Martin, M.A., Ph.D.)—Show that the value of the integral $\int_0^1 [(1 + e^x) E(e, 1) - (1 - e^x) F(e, 1)] dx$,

where $F(e, 1)$ and $E(e, 1)$ denote complete elliptic functions of the first and second kinds, is 72445 93

7232. (By R. F. Scott, M.A.)—If (a, b, c) stand for the straight line $ax + by + cz = 0$, show that the six lines obtained by permutating a, b, c in (a, b, c) in all possible ways, all touch the conic

$$\frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{2(yz + zx + xy)}{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab} = 0 \quad \dots\dots\dots 75$$

7233. (By G. S. Carr, B.A.)—If two non-intersecting circles, having radii a, b , centres A, B, and distance between the centres c , be inverted into concentric circles with radii α, β , and centre O; show that, if we denote

$$(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)^{\frac{1}{2}}$$

by S, we have

$$AO = \frac{a^2 - b^2 - c^2 - S}{2c} + \frac{k^2c}{S}, \quad \alpha = \frac{2k^2c^2a}{S(a^2 - b^2 + c^2 - S)}, \quad \beta = \frac{\pm 2k^2c^2b}{S(a^2 - b^2 - c^2 - S)},$$

a being $> b$ and the + or - sign being read according as the smallest circle is within or without the largest 63

7235. (By C. Leudesdorf, M.A.)—Each of three circles (radii ρ_1, ρ_2, ρ_3) touches two sides of a plane triangle ABC and the nine-point circle, having external contact with the latter; if R, r be the radii of the circumscribed and inscribed circles of ABC, and Δ its area, prove that

$$(r - \rho_1)(r - \rho_2)(r - \rho_3) = 2R^2r^2\Delta^{-2} \dots\dots\dots 96$$

7238. (By W. J. C. Sharp, M.A.)—Prove that the circumscribed circle of a triangle bisects the lines $OO_1, OO_2, OO_3, O_2O_3, O_3O_1, O_1O_2$, where O, O_1, O_2, O_3 are the centres of the inscribed and circumscribed circles 77

7237. (By W. J. C. Sharp, M.A.)—If cones be constructed having their vertices at two given points and enveloping a given quadric; prove that (1) their common curve will consist of two conics lying on two planes which pass through the intersection of the polar planes of the given points; and (2) if these points lie on a quadric concentric, similar, and similarly situated to the given one, one of the planes will pass through the centre, and the other will be the polar plane of the middle point of the line joining the given points 84

7239. (By G. Turriff, M.A.)—The angle B and the exterior angle A of a triangle ABC are bisected by straight lines meeting in G. With centre C and radius equal to OG, a circle is drawn cutting GA, GB, or these produced, in D, E. Prove that DE is parallel to AB 76

7241. (By A. McMurchy, B.A.)—Show that, if m, n, p, q be positive

integers, the limiting value, when $x = y = z = a$, of the fraction

$$\frac{x^m (y^m - z^n) + y^m (z^n - x^n) + z^m (x^n - y^n)}{x^p (y^q - z^q) + y^p (z^q - x^q) + z^p (x^q - y^q)}, \text{ is } \frac{mn(m-n)}{pq(p-q)} a^{m+n-p-q} \dots 76$$

7242. (By S. Tebay, M.A.)—*Ring-leaping*. The gymnast will know the meaning of this phrase. Two hand-rings are suspended from a beam by cords. The leaper sets himself in motion by swinging to and fro, and when this has become sufficiently lively he quits the rings and alights on the ground. Find the greatest distance he can leap 101

7246. (By Capt. MacMahon, R.A.)—From any point P in the side BC of the triangle ABC, lines PD, PE are drawn parallel to AC and AB, meeting those sides in D and E; prove that the envelope of DE is a parabola, whose focus is at a distance from A equal to $bc[2(b^2 + c^2) - a^2]^{-1}$. 74

7249. (By Professor Sylvester, F.R.S.)—If

$$1 - \frac{1}{1-c} y + \frac{c}{(1-c)(1-c^2)} y^2 + \dots \pm \frac{c^{1(n^2-n)}}{(1-c)(1-c^2) \dots (1-c^n)} y^n + \dots = 0,$$

prove that (1) the sum of the i^{th} powers of the several values of $\frac{1}{y}$ is $\frac{1}{1-c^i}$; and (2) the sum of the i^{th} homogeneous products of the same values is

$$\frac{1}{(1-c)(1-c^2) \dots (1-c^i)}.$$

[If $i = 2$, the first sum is $\left(\frac{1}{1-c}\right)^2 - 2 \frac{c}{(1-c)(1-c^2)}$, which is $\frac{1}{1-c^2}$, and the second sum is $\left(\frac{1}{1-c}\right)^2 - \frac{1}{(1-c)(1-c^2)}$, which is $\frac{1}{(1-c)(1-c^2)}$.] 50

7255. (By Professor Martin, M.A., Ph.D.)—An urn contains 12 balls, of which either 5, or 6, or 7, or 8, are white (but which one of these numbers is not known), and the rest are black; if a ball is drawn at random, find the chance that it is white 51

7257. (By Professor Malet, F.R.S.)—A cone of the second degree intersects a given sphere in two planes, one fixed, and the other touching in every position another fixed sphere concentric with the first; prove that the locus of the vertex of the cone is a pair of spheres 80

7258. (By the Editor.)—If a square board, of side a , be supported in a horizontal position by two equal loops of string, of length λ , that pass under opposite corners, and are suspended from a fixed hook; prove that the tensions of the strings are

$$\frac{W}{4} \cdot \frac{\lambda - a\sqrt{2}}{(\lambda^2 - 2\sqrt{2}a\lambda)^{\frac{1}{2}}} \dots 115$$

7259. (By Professor Minchin, M.A.)—A point P describes an orbit, with acceleration f , directed always to a fixed centre O; prove that the acceleration of the corresponding point P', of the Hodograph, is in general, non-central, and makes with the radius vector OP' an angle

$$\chi \text{ given by the equation } \tan \chi = \frac{\frac{d}{d\theta} \log \frac{f}{r}}{1 - \cot \phi \frac{d}{d\theta} \log f}, \text{ where } \phi \text{ is the angle}$$

between OP and the tangent at P, $r = OP$, and θ is the polar angle of r
 d 52

7262. (By E. B. Elliott, M.A.)—Two bicircular curves of the fourth class have the foci of a given conic Σ for double foci, and have each a pair of tangents to Σ for double tangents. Prove that every common tangent of the two, except the tangent at the circular points, passes through one or other of two fixed points dividing harmonically the distance between the intersections of the two pairs of tangents to Σ 103

7263. (By J. J. Walker, M.A.)—Show that the principal axes through any point (O) of a homogeneous solid, with respect to which the bounding surface (S) is convex, are also principal axes for the shell (S) supposed to have at each point a surface density proportional to the perpendicular from (O) on the tangent plane at that point..... 62

7266. (By Dr. Macfarlane, F.R.S.E.)—Find the number of different possible states of the poll, under the cumulative system of voting, when there are p candidates, q seats, and r electors 94

7267. (By C. Morgan, B.A.)—If arcs of great circles be drawn from the angles A and B of a spherical triangle perpendicular to the opposite sides; prove that, if $\tan \alpha \tan \alpha' = \tan \beta \tan \beta'$, where $\alpha, \alpha', \beta, \beta'$ are the segments into which the sides are divided, the triangle must be either isosceles or right-angled 96

7269. (By Belle Easton.)—One of two bags contains 10 sovereigns, and the other 10 shillings. One coin is taken out of each and placed in the other. This is repeated ten times. Show that now the expectations of these bags are $105 \pm 95 (\frac{1}{2})^n$ 71

7272. (By R. F. Scott, M.A.)—If S and H are the foci of a system of confocal conics, and on SH as diameter a circle is drawn; prove (1) that the locus of the extremities of the diameters conjugate to those drawn to the points where the circle meets the conics of the system, is a lemniscate of Bernoulli; and (2) find what the locus becomes for a circle concentric with the given circle..... 82

7277. (By Professor Sylvester, F.R.S.)—If S_k denotes the sum of the k -ary products of $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i-1}$, prove that the coefficient of any positive $(1+k)^{\text{th}}$ power of $\frac{1}{x}$, in the development of $\frac{1}{(1-e^{-x})^i}$ in a series of ascending powers of x , is $\Pi k \cdot S_k$ 74

7279. (By Professor Hudson, M.A.)—On a given base AB a right-angled spherical triangle ACB is drawn; on CA, CB, CP, CQ are taken each equal to a quadrant; find the loci of P and Q 79

7280. (By Professor Malet, F.R.S.)—If any three conjugate semi-diameters of the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = \frac{1}{k^2}, \quad \frac{x^2}{a''^2} + \frac{y^2}{b''^2} + \frac{z^2}{c''^2} = \frac{1}{k'^2},$$

meet the three surfaces respectively in the points P, Q, R, prove that the envelope of the plane PQR is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{1+k^2+k'^2} \quad \dots \quad 73$$

7283. (By Professor Martin, M.A., Ph.D.)—From a full cask of

wine a quantity is drawn off at random, and the cask filled up with water. Find the chance that, after n such operations, the cask contains more wine than water 114

7285. (By the Editor.)—From every point in a given curve a straight line is drawn, making with the ordinate on one side the same angle that the tangent makes with the ordinate on the other side: find (1) the locus of the intersections of these lines; and (2) prove that, if the given curve is the common parabola, the locus will be a parabola whose latus-rectum is triple that of the first 72

7286. (By W. J. C. Sharp, M.A.)—Prove that perpendiculars drawn through the middle points of the sides of a triangle to the inner bisectors of the angles are the radical axes of the inscribed and the several escribed circles 79

7289. (By R. Warburton-White, B.A.)—Show that the nine-point circle touches the inscribed and escribed circles of a triangle 99

7299. (By R. Knowles.)—P is any point on an ellipse of which S and H are the foci; circles are drawn on SP and HP as diameters; show that (1) the loci of their centres are two equal ellipses intersecting on the minor axis of the given ellipse; (2) the circles touch the auxiliary circle; (3) the line joining the points of contact touches the ellipse at P 73

7302. (By Professor Townsend, F.R.S.)—If two orthogonal systems of lines in the plane of an incompressible lamina be the potential and displacement line-systems of a possible irrotational strain of the lamina; show that—

(a) The two inverse systems of lines with respect to any arbitrary circle, real or imaginary, in the plane will be those of another possible irrotational strain of the lamina.

(b) At all pairs of corresponding points of the two inverse strains, the ratios of the two actual displacements and of the two principal dilatations are proportional respectively to those of the simple and those of the squared inverse distances from the centre of the circle 97

7303. (By Professor Morel.)—Si A, B, C, a , b , c sont les éléments d'un triangle, et ϕ un angle satisfaisant à la relation

$$\frac{b^2 + c^2 - 2bc \cos(A - 2\phi)}{b^2 + c^2 - 2bc \cos A} = \frac{c^2 + a^2 - 2ca \cos(B - 2\phi)}{c^2 + a^2 - 2ca \cos C};$$

(1) l'un quelconque de ces rapports est aussi égal à

$$\frac{a^2 + b^2 - 2ab \cos(C - 2\phi)}{a^2 + b^2 - 2ab \cos C};$$

et en outre (2) on a, en supprimant la solution $\phi = 0$, la relation

$$\cot \phi = \cot A + \cot B + \cot C \dots\dots\dots 108$$

7307. (By Professor Wolstenholme, M.A.)—A given rigid body, of mass m , is kinetically equivalent to a system of four equal particles (each of mass $\frac{1}{4}m$) placed at the corners of any maximum tetrahedron inscribed

in the ellipsoid
$$\frac{x^2}{B+C-A} + \frac{y^2}{C+A-B} + \frac{z^2}{A+B-C} = \frac{3}{2},$$

where MA, MB, MC are the moments of inertia about the principal axes at the centroid, which are also the axes of coordinates. If the notation,

MA = $\Sigma (mx^2)$, MB = $\Sigma (my^2)$, MC = $\Sigma (mz^2)$, be used, the equation of the ellipsoid will be $\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 3$ 105

7309. (By W. J. C. Sharp, M.A. Suggested by Question 7249.)—If S_r denote the value of the sum of the r^{th} homogeneous products of any quantities, and s_r the sum of the r^{th} powers of the same quantities; show that

$$S_1 = s_1, \quad 2S_2 = S_1 s_1 + s_2, \quad 3S_3 = S_2 s_1 + S_1 s_2 + s_3,$$

and

$$rS_r = S_{r-1}s_1 + S_{r-2}s_2 + S_{r-3}s_3 + \dots + s_r \dots\dots\dots 95$$

7311. (By J. J. Walker, M.A.)—If a point P describes a circumference of radius a with a velocity varying as the square of the distance (v) from any fixed point in its plane, show that the acceleration makes with the circumference at P an angle ψ determined by

$$\cot \psi = -2a \cos \phi + r,$$

ϕ being the angle which r makes with the circumference at P; and that the acceleration varies as $r^3 \cos \phi + \cos \psi$ 93

7312. (By R. Tucker, M.A.)—Along the axis of a parabola ($y^2 = 9ax$) distances AK = KL = LM = a are measured from the vertex (A), and the double ordinates BKB', CLC', DMD' are drawn; on DD', lengths MR = MS = a are taken, and lines RP, SP, through any point P on CC', meet BB' in T, T'. Prove that perpendiculars to ST, RT', at T, T', meet on the parabola. [This theorem is susceptible of considerable generalization by assigning different lengths to AK, KL, LM, MR, MS.] 97

7313. (By C. Leudesdorf, M.A.)—Prove that the equations

$$\frac{ad - bc}{2(ac - b^2)} = \frac{ae + 2bd - 3c^2}{3(ad - bc)} = \frac{3(bc - cd)}{ae + 2bd - 3c^2} = \frac{2(ce - d^2)}{be - cd}$$

are equivalent to $ae + 3c^2 - 4bd = 0$, $ace + 2bcd - ad^2 - eb^2 - c^3 = 0$ 91

7315. (By W. S. McCay, M.A.)—If XX', YY', ZZ' be the diagonals of a quadrilateral circumscribed to a circle, L, M, N their mid-points, and O the centre of the circle; prove that

$$\frac{OX \cdot OX'}{OL} = \frac{OY \cdot OY'}{OM} = \frac{OZ \cdot OZ'}{ON} \dots\dots\dots 88$$

7320. (By R. Knowles, B.A.) — If n_r be the coefficient of x^r in the expansion of $(1+x)^n$, prove that

$$u = n_1 - \frac{2^2-1}{2} n_2 + \frac{2^3-1}{3} n_3 - \dots + (-1)^{n-1} \frac{2^n-1}{n} n_n \\ = 1 - \frac{1}{2} + \frac{1}{3} - \dots + (-1)^n \frac{1}{n} \dots\dots\dots 102$$

7322. (By Professor Sylvester, F.R.S.)—Prove that the number of sequences of consecutive numbers (a "sequence" of one term not excluded) having the sum n , is equal to the number of odd divisors of n .

Ex.—The odd divisors of 9 are 1, 3, 9, and 9 is represented by the three sequences 2, 3, 4; 4, 5; 9. The odd divisors of 15 are 1, 3, 5, 15, and 15 is represented by the four sequences 1, 2, 3, 4, 5; 4, 5, 6; 7, 8; 15.

[The extension of the theorem is as follows: n can be represented by pairs of sequences of consecutive numbers having no common term in as many ways as there are ways of satisfying the equation in positive integers $x(2y+1) + z(2t+1) = n$. Again, n can be represented by as many triads

of sequences having no common term as there are solutions or positive integers of the equation $x(2y+1) + z(2t+1) + u(2v+1) = n$, and so on.] 122

7324. (By Professor Wolstenholme, M.A.) — A triangle $A'B'C'$, whose angles are given, circumscribes a given triangle ABC ; prove that there will be two such triangles of maximum area; and, if these areas be Δ_1 , Δ_2 , and R the radius of the circle ABC ,

$$\Delta_1 \sim \Delta_2 = 8R^2 \sin A \sin B \sin C,$$

$$\Delta_1 + \Delta_2 = 4R^2 (\sin^2 A \cot A' + \sin^2 B \cot B' + \sin^2 C \cot C').$$

[If the *sign* of the areas be reckoned according to the usual rule (that $A'B'C'$ is positive when, in going round the triangle in the order A' , B' , C' , the area is on the left hand), the smaller of these will be negative, and with this convention $\Delta_1 + \Delta_2 = 8R^2 \sin A \sin B \sin C$,

$$\Delta_1 - \Delta_2 = 4R^2 (\sin^2 A \cot A' + \dots + \dots).]$$
 124

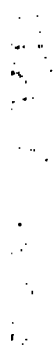
7327. (By Professor Wolstenholme, M.A.) — Prove that (1) the maximum value of $f(x) \equiv (1+x)^{\frac{1}{2}(1+x)} (1-x)^{\frac{1}{2}(1-x)} + (1-x^2)^{\frac{1}{2}} - 2$ lies between $\cdot 093511$ and $\cdot 093512$; and (2) the corresponding value of x is $\cdot 9653863$ very nearly. 109

7328. (By the Editor.) — Show that the locus of the vertex of a triangle on a given base, when the centre of the inscribed square moves on a given conic, is another conic 117

7330. (By the Editor.) — Prove that the area of a triangle whose sides are roots of the equation $f(x) \equiv x^3 + px^2 + qx + r = 0$, is

$$\Delta = \frac{1}{2} p^{\frac{1}{2}} (4pq - p^3 - 8r)^{\frac{1}{2}} \dots\dots\dots 111$$

7338. (By S. Tebay, B.A.) — The parallel sides of a trapezoid are $m^2 + 2mn - n^2$ and $n^2 + 2mn - m^2$; show that it is divided into two equal trapezoids by the line $m^2 + n^2$ 86



MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

7074. (By Professor MANNHEIM.)—On a n points $f, f', f'', \dots$ de la courbe (m) , lieu des points tels que m , pour lequel on a $mf \cdot r \cdot mf' \cdot r \cdot mf'' \cdot r \dots = \text{const.}$ Soient μ le centre de courbure de m , pour le point m ; p le point de rencontre de la normale en m à (m) , et de la perpendiculaire élevée de f à fm ; $p', p'', \dots$ des points analogues à p pour $f, f', f'', \dots$

Démontrer que $\frac{1}{m\mu} \sum \left(\frac{1}{mf^2} \right) = 2 \sum \left(\frac{1}{mp^2} \right) - \sum \left(\frac{1}{mf^2} \right)$.

Solution by W. J. C. SHARP, M.A.; CHRISTINE LADD, B.A.; and others.

Denote the distances $mf, mf', \dots$ by $r, r', \dots$, the angles which these make with a prime radius by $\theta, \theta', \dots$, and those they make with the tangent by $\psi, \psi', \dots$; then $r, r', \dots = c$; hence, differentiating,

$$\sum \frac{dr}{r} = 0, \quad \text{or} \quad \sum \frac{r \psi}{r} = 0,$$

since $dr = \cos \psi \, ds$. Differentiating again,

$$\sum \frac{\cos \psi}{r^2} \cdot \frac{dr}{ds} + \sum \frac{\sin \psi}{r} \cdot \frac{d\psi}{ds} = 0, \quad \therefore \sum \frac{\cos^2 \psi}{r^2} = - \sum \frac{\sin \psi}{r} \cdot \frac{d\psi}{ds};$$

$$\text{and} \quad \sum \frac{\sin^2 \psi}{r^2} = \sum \frac{\sin \psi}{r} \cdot \frac{d\theta}{ds}, \quad \because \sin \psi = r \frac{d\theta}{ds};$$

$$\text{therefore} \quad \sum \frac{1}{r^2} = \sum \frac{\sin \psi}{r} \left\{ \frac{d\theta}{ds} - \frac{d\psi}{ds} \right\} = 2 \sum \frac{\sin^2 \psi}{r^2} - \frac{d\phi}{ds} \sum \frac{\sin \psi}{r},$$

since

$$\phi = \theta + \psi;$$

$$\text{therefore} \quad \sum \left(\frac{1}{mf^2} \right) = 2 \sum \left(\frac{1}{mp^2} \right) - \frac{1}{m\mu} \sum \left(\frac{1}{mp} \right),$$

therefore, &c.

The method here employed may be used with advantage when a vector equation to a curve is given.

7167. (By Professor MINCHIN, M.A.)—A rigid body has uniplanar motion; find the locus, at any instant, of points the directions of whose velocities make a given angle with the directions of their accelerations.

Solution by J. J. WALKER, M.A.

If O is the instantaneous centre, and J the point of no acceleration for any plane section parallel to the given direction of motion, then the direction of acceleration of a point P in the section makes with JP an angle $\phi = \tan^{-1} \omega + \omega^2$ (Quest. 6539, *Reprint*, xxxv., p. 45), which at any instant is the same for all points; while the direction of velocity of P is perpendicular to OP; hence (a being the angle between these directions) the angle OPJ = $\phi + a - \frac{1}{2}\pi$, and is given. The locus of P is therefore a circumference at the points of which the line OJ subtends a given angle; and the locus of points throughout the body satisfying the given condition is a right cylinder.

It is of interest to observe that, as appears from Notes of the late Professor CLIFFORD's Lectures on "Quaternions" (*Mathematical Papers*, p. 502), delivered in University College, London, in the year 1877, attention was drawn in them to the existence of the point of no acceleration, in the case of uniplanar motion.

6768. (By Professor CROFTON, F.R.S.)—Prove that

$$f(x+hD) \cdot 1 \equiv f(x) + \frac{h}{2} f''(x) + \frac{h^2}{2 \cdot 4} f'''(x) + \&c.$$

Solution by G. F. WALKER, M.A.; Professor SEITZ, M.A.; and others.

We have $(x+hD) \cdot 1 = x$, $(x+hD)^2 \cdot 1 = x^2 + \frac{1}{2}h \cdot 2$.

Assume $(x+hD)^n \cdot 1 = x^n + \dots + \frac{h^r}{2 \cdot 4 \dots 2r} n(n-1) \dots (n-2r+1) x^{n-2r}$;

then $(x+hD)^{n+1} \cdot 1 = \dots + \frac{h^r}{2 \cdot 4 \dots 2r} n(n-1) \dots (n-2r+1) x^{n-2r+1}$

$$+ \frac{h^r}{2 \cdot 4 \dots 2(r-1)} n(n-1) \dots (n-2r+3)(n-2r+2) x^{n-2r+1} \\ = \dots + \frac{h^r}{2 \cdot 4 \dots 2r} (n+1)n(n-1) \dots (n-2r+2) x^{n-2r+1}.$$

Hence, if the formula is true for n , it is true for $n+1$. But it is true for $n=1$ and $n=2$, and therefore generally. Therefore

$$(x+hD)^n \cdot 1 = x^n + \dots + \frac{h^r}{2 \cdot 4 \dots 2r} \frac{d^{2r}}{dx^{2r}} \cdot x^n.$$

But

$$f(x) = A_0 + B_1x + \dots + A_nx^n + \dots;$$

hence, generally,

$$f(x+hD) \cdot 1 = f(x) + \frac{h}{2} f''(x) + \dots + \frac{h^r}{2 \cdot 4 \dots 2r} f^{(2r)}(x) + \dots$$

7101. (By Professor TOWNSEND, F.R.S.)—A cubic surface being supposed to have a nodal point; show that, of the three triple tangent planes intersecting in any one of its twenty-seven superficial lines, the three points of contact not on the line connect with the nodal point by a triad of conjugate diameters of the nodal cone.

Solution by the PROPOSER.

Denoting by O the nodal point, and by I the superficial line of the cubic; by X, Y, Z the three points of contact not on I of the three triple tangent planes to the surface which intersect in I; by P, Q, R the three points of intersection of the three nodal planes YOZ, ZOX, XOY with the line I; and by LL', MM', NN' their three angles of intersection with the nodal cone at O; then, since, for the three nodal sections of the surface by the three aforesaid planes, the three pairs of lines PY and PZ, QZ and QX, RX and RY are three pairs of tangents intersecting at the three points P, Q, R on the three curves, therefore, by a familiar property of plane nodal cubics, the three pairs of lines OY and OZ, OZ and OX, OX and OY divide harmonically the three angles LL', MM', NN', and therefore, &c.

7099. (By the EDITOR.)—Sum (1) the series

$$\frac{\sin^3 \theta}{1^4} - \frac{\sin^3 3\theta}{3^4} + \frac{\sin^3 5\theta}{5^4} - \&c.$$

between the limits $\theta = 0$, $\theta = 2\pi$; and $\theta = \frac{1}{3}\pi$, $\theta = \frac{1}{3}\pi$; and hence

show (2) that $\pi^4 = \frac{3 \cdot 2^{1/2}}{11} \left(\frac{1}{1^4} - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \&c. \right)$.

Solution by Prof. WOLSTENHOLME, M.A.; Prof. NASH, M.A.; and others.

Transforming each term by the equation $\sin^3 x = \frac{1}{4} (3 \sin x - \sin 3x)$, the given series becomes $\frac{3}{4}S_1 - \frac{1}{4}S_3$, where

$$S_1 = \sin \theta - \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta - \dots, \quad S_3 = \sin 3\theta - \frac{1}{3} \sin 9\theta + \frac{1}{5} \sin 15\theta - \dots$$

Also, for values of ϕ between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$,

$$\frac{1}{2}\pi = \cos \phi - \frac{1}{3} \cos 3\phi + \frac{1}{5} \cos 5\phi - \dots,$$

and, integrating this three times, we shall have, with the same limits

$$\frac{1}{8}\pi\phi \left(\frac{1}{2}\pi^2 - \frac{1}{2}\phi^2 \right) = \sin \phi - \frac{\sin 3\phi}{3^4} + \frac{\sin 5\phi}{5^4} - \frac{\sin 7\phi}{7^4} \dots \dots \dots (A).$$

Hence, for a value of ϕ between $r\pi - \frac{1}{2}\pi$ and $r\pi + \frac{1}{2}\pi$, r any integer,

$$(-1)^r \frac{1}{8}\pi \left(\phi - r\pi \right) \left[\frac{1}{2}\pi^2 - \frac{1}{2}(\phi - r\pi)^2 \right] = \sin \phi - \frac{\sin 3\phi}{3^4} + \frac{\sin 5\phi}{5^4} - \frac{\sin 7\phi}{7^4} + \dots$$

The value of the proposed infinite series will therefore be

(1) from $\theta = -\frac{1}{2}\pi$ to $\theta = \frac{1}{2}\pi$;

$$\frac{1}{8}\pi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left(\frac{1}{2}\pi^2 - \frac{1}{2}\theta^2 \right) d\theta - \frac{1}{8}\pi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 3\theta \left(\frac{1}{2}\pi^2 - 3\theta^2 \right) d\theta = \frac{1}{2}\pi^3;$$

- (2) $\theta = \frac{1}{4}\pi$ to $\theta = \frac{1}{2}\pi$;
 $\frac{1}{2} \frac{1}{\pi} \theta \left(\frac{1}{4}\pi^2 - \frac{1}{2}\theta^2 \right) + \frac{1}{4} \frac{1}{\pi} (3\theta - \pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - \pi)^2 \right]$;
(3) $\theta = \frac{1}{2}\pi$ to $\theta = \frac{3}{4}\pi$;
 $-\frac{3}{4} \frac{1}{\pi} (\theta - \pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - \pi)^2 \right] - \frac{1}{4} \frac{1}{\pi} (3\theta - 2\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - 2\pi)^2 \right]$;
(4) $\theta = \frac{3}{4}\pi$ to $\theta = \pi$;
 $-\frac{3}{4} \frac{1}{\pi} (\theta - \pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - \pi)^2 \right] + \frac{1}{4} \frac{1}{\pi} (3\theta - 3\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - 3\pi)^2 \right]$
 $= -\frac{1}{4}\pi (\theta - \pi)^3$;
(5) $\theta = \frac{5}{4}\pi$ to $\theta = \frac{3}{2}\pi$;
 $-\frac{3}{4} \frac{1}{\pi} (\theta - \pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - \pi)^2 \right] - \frac{1}{4} \frac{1}{\pi} (3\theta - 4\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - 4\pi)^2 \right]$;
(6) $\theta = \frac{3}{2}\pi$ to $\theta = \frac{7}{4}\pi$;
 $\frac{1}{4} \frac{1}{\pi} (\theta - 2\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - 2\pi)^2 \right] + \frac{1}{4} \frac{1}{\pi} (3\theta - 5\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - 5\pi)^2 \right]$;
(7) $\theta = \frac{7}{4}\pi$ to $\theta = \frac{5}{2}\pi$;
 $\frac{1}{4} \frac{1}{\pi} (\theta - 2\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - 2\pi)^2 \right] - \frac{1}{4} \frac{1}{\pi} (3\theta - 6\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - 6\pi)^2 \right]$
 $= \frac{1}{4}\pi (\theta - 2\pi)^3$;

and, in general, if θ lie between $(2n-1)\frac{1}{4}\pi$ and $(2n+1)\frac{1}{4}\pi$, the sum will be

$$\frac{1}{4}\pi \left\{ (-1)^n 3(\theta - r\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - r\pi)^2 \right] - (-1)^n (3\theta - n\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - n\pi)^2 \right] \right\},$$

where r is $\frac{1}{2}(n+1)$, or the integer next less.

If, in the equation (A), we put $\phi = \frac{1}{4}\pi$, we get

$$\frac{1}{4}\pi \pi^4 \left(\frac{1}{4} - \frac{1}{48} \right) = \frac{1}{\sqrt{2}} \left\{ \frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \dots \right\},$$

or
$$\pi^4 = \frac{3 \times 2^{1/2}}{11} \left\{ \frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \dots \right\};$$

which is also obtained by putting $\theta = \frac{1}{4}\pi$ in the proposed series, and taking the value (2); $\theta = \frac{1}{2}\pi$ and (3); $\theta = \frac{3}{4}\pi$ and (5); or $\theta = \frac{5}{4}\pi$ and (6).

[In the same way, using the identity,

$$\sin^5 \phi = \frac{1}{16} (10 \sin \phi - 5 \sin 3\phi + \sin 5\phi),$$

we can sum
$$\frac{\sin^5 \theta}{1^4} - \frac{\sin^5 3\theta}{3^4} + \frac{\sin^5 5\theta}{5^4} - \frac{\sin^5 7\theta}{7^4} + \dots,$$

and for the first range, from $-\frac{1}{16}\pi$ to $\frac{1}{16}\pi$, the sum is zero; and will, in general, be

where
$$\begin{aligned} S_1 &= (-1)^p (\theta - p\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (\theta - p\pi)^2 \right], \\ S_3 &= (-1)^q (3\theta - q\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (3\theta - q\pi)^2 \right], \\ S_5 &= (-1)^r (5\theta - r\pi) \left[\frac{1}{4}\pi^2 - \frac{1}{2} (5\theta - r\pi)^2 \right]; \end{aligned}$$

where p, q, r are integers such that each of the angles $\theta - p\pi, 3\theta - q\pi, 5\theta - r\pi$ lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$. It is obvious that, in the same way, the sum of the infinite series

$$\frac{\sin^n \left(\frac{\theta}{n} \right)}{1^{2m}} - \frac{\sin^n \left(\frac{3\theta}{n} \right)}{3^{2m}} + \frac{\sin^n \left(\frac{5\theta}{n} \right)}{5^{2m}} - \dots,$$

where n is any odd integer, and m an integer less than $\frac{1}{2}n$, will be zero from $\theta = -\frac{1}{4}\pi$ to $\theta = \frac{1}{4}\pi$; and

$$\frac{\sin^{2m-1} \theta}{1^{2m}} - \frac{\sin^{2m-1} (3\theta)}{3^{2m}} + \dots = \frac{1}{4}\pi \theta^{2m-1},$$

from $\theta = -\frac{\pi}{4m-2}$ to $\theta = \frac{\pi}{4m-2}$.]

7066. (By Dr. MACFARLANE, F.R.S.E.)—Find the number of ways in which an elector may vote, under the cumulative system of voting, when there are p candidates for q seats. [Applied to the recent election of a School Board for Dundee, Dr. MACFARLANE'S results show that the number of possible ways of voting was 43 times the population of the globe.]

Solution by the PROPOSER.

Under the cumulative system of voting, an elector may give any number r of votes not greater than q , the number of seats; and he may distribute the r votes in any manner he pleases among the p candidates. The number of ways in which he can distribute r votes among p candidates is the same as the number of homogeneous products of r dimensions which can be formed out of p symbols; hence it is $\frac{p+r-1!}{p-1! r!}$. Hence the

total number of ways in which the elector may vote is $\frac{p+q!}{p! q!}$; and, as any mathematician will at once see, the electors will, in general, have an ample choice.

7047. (By R. RUSSELL, B.A.)—Show (1) generally how to solve

$$(a_1 b_1 c_1 f_1 g_1 h_1) (x, y, z)^2 = l_1, \quad (a_2 b_2 c_2 f_2 g_2 h_2) (x, y, z)^2 = l_2, \\ (a_3 b_3 c_3 f_3 g_3 h_3) (x, y, z)^2 = l_3;$$

and (2) prove that the solution of

$$x^2 + yz = a, \quad y^2 + zx = b, \quad z^2 + xy = c,$$

depends on the solution of $u^3 + v^3 + w^3 = 5uvw$, where

$$u = (b-c) \left(1 + \frac{\lambda}{a}\right), \quad v = (c-a) \left(1 + \frac{\lambda}{b}\right), \quad w = (a-b) \left(1 + \frac{\lambda}{c}\right).$$

[This is connected with the EDITOR'S Questions 6928 and 6972.]

Solution by the PROPOSER; D. EASTWOOD, M.A.; and others.

$$S_1 \equiv (a_1 b_1 c_1 f_1 g_1 h_1) (xyz)^2 = l_1, \quad S_2 \equiv (a_2 b_2 c_2 f_2 g_2 h_2) (xyz)^2 = l_2,$$

$$S_3 \equiv (a_3 b_3 c_3 f_3 g_3 h_3) (xyz)^2 = l_3;$$

$$\text{therefore} \quad U \equiv (l_2 - l_3) S_1 + (l_3 - l_1) S_2 + (l_1 - l_2) S_3 = 0,$$

$$V \equiv \frac{l_2 - l_3}{l_1} S_1 + \frac{l_3 - l_1}{l_2} S_2 + \frac{l_1 - l_2}{l_3} S_3 = 0.$$

The determination of λ such that $U + \lambda V$ may represent two linear factors depends on a cubic of course.

Hence, in solving this cubic, we obtain a line as relation between x, y, z . By means of which, and any two of the original equations, we can at once solve for x, y , and z .

In the particular case of $x^2 + yz = a$, $y^2 + zx = b$, $z^2 + xy = c$,

$$U + \lambda V \equiv (x^2 + yz)(b - c) \left(1 + \frac{\lambda}{a}\right) + (y^2 + zx)(c - a) \left(1 + \frac{\lambda}{b}\right) \\ + (z^2 + xy)(a - b) \left(1 + \frac{\lambda}{c}\right) = 0,$$

or $ux^2 + vy^2 + wz^2 + uyz + vzx + wxy = 0$, therefore $u^3 + v^3 + w^3 = 5uvw$,

where $u = (b - c) \left(1 + \frac{\lambda}{a}\right)$, $v = (c - a) \left(1 + \frac{\lambda}{b}\right)$, $w = (a - b) \left(1 + \frac{\lambda}{c}\right)$.

7125. (By Professor WOLSTENHOLME, M.A.)—Prove that, if

$$y = \frac{1}{e^x - 1}, \quad \frac{d^n y}{dx^n} = (-1)^n \left\{ \frac{A_1}{e^x - 1} + \frac{A_2}{(e^x - 1)^2} + \dots + \frac{A_{n+1}}{(e^x - 1)^{n+1}} \right\},$$

where

$$A_p = \Delta^p 0^n + \Delta^{p-1} 0^n.$$

Solution by W. J. C. SHARP, M.A.; Professor MATZ, M.A.; and others.

If $y = (e^x - 1)^{-1}$, since this may be written

$$e^{-x} + e^{-2x} + e^{-3x} + \dots + e^{-rx} + \dots,$$

we have $\frac{d^n y}{dx^n} = (-1)^n (e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \dots + r^n e^{-rx} + \dots)$.

Now $\frac{1}{x} + \frac{2^n}{x^2} + \frac{3^n}{x^3} + \dots$ to $\infty \equiv \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \dots + \frac{A_{n+1}}{(x-1)^{n+1}}$,

if $A_1 = 1^n$, $A_1 + A_2 = 2^n$, $A_1 + 2A_2 + A_3 = 3^n$, &c.; and these equations may be written $A_1 = 1^n$, $A_1 + A_2 = (1 + \Delta) 1^n$, $A_1 + 2A_2 + A_3 = (1 + \Delta)^2 1^n$, and so on, from which form it is obvious that

$$A_p = \Delta^{p-1} 1^n = \Delta^{p-1} 0^n + \Delta^p 0^n.$$

[It is easy to prove that

$$A_1 + A_3 + A_5 + \dots = A_2 + A_4 + A_6 + \dots \\ = 2^n A_{n+1} - 2^{n-1} A_n + 2^{n-2} A_{n-1} - \dots + (-1)^n A_1, \\ = \frac{1}{2} \left(1^n + \frac{2^n}{2} + \frac{3^n}{2^2} + \dots + \frac{r^n}{2^{r-1}} + \dots \text{to } \infty \right),$$

the last series being always convergent.]

7084. (By J. GRIFFITHS, M.A.)—Find the locus of intersection of three mutually perpendicular planes to an elliptic paraboloid.

Solution by D. EDWARDS; J. O'REGAN; and others.

Let the paraboloid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$, and a tangent plane thereto

$$z_1 x + m_1 y + n_1 z = p \equiv -\frac{a^2 l_1^2 + b^2 m_1^2}{2cn_1}, \text{ or } n_1 l_1 x + m_1 n_1 y + n_1^2 z + a^2 l_1^2 + b^2 m_1^2 = 0;$$

whence, since $\Sigma n_1 l_1 = 0$, $\Sigma m_1 n_1 = 0$, $\Sigma n_1^2 = \Sigma l_1^2 = \Sigma m_1^2 = 1$, we have, by addition for the required locus, the plane $z + \frac{a^2 + b^2}{2c} = 0$.

7179. (By the Rev. G. RICHARDSON, M.A.)—If AOD be the perpendicular from A on the opposite side of the triangle ABC, O being the orthocentre, I the centre of the inscribed circle, r its radius; prove that $IO^2 = 2r^2 - AO \cdot OD$.

Solution by W. H. BLYTHE, M.A.; R. KNOWLES, B.A.; and others.

Here we readily find $AD = c \sin B$, $OD = b \cos C \cdot \cot B$,
 therefore $AO = AD - OD = b \cos A \operatorname{cosec} B$;
 $\therefore AO \cdot OD = b^2 \operatorname{cosec}^2 B \cdot \cos A \cos B \cos C = 4R^2 \cos A \cos B \cos C$;
 also (see solution of Quest. 6313),
 $OI^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$,
 therefore $\quad \quad \quad = 2r^2 - AO \cdot OD$.

7188. (By Dr. CURTIS.)—Prove that the centre of gravity of the area included by a loop of the curve $r = a \cos n\theta$ is at a distance from the origin

$$= 16an^4 \sin\left(\frac{n+1}{2n} \pi\right) + \{(n^2-1)[(3n)^2-1]\pi\}.$$

[The cases $n = 2$, $n = 3$ are exercises in TODHUNTER'S *Statics*.]

Solution by W. M. COATES, B.A.; W. H. BLYTHE, B.A.; and others.

In the curve $r = a \cos n\theta$, $r = a$ for $\theta = 0$ and $r = 0$ for $\theta = \frac{\pi}{2n}$, and as loop is symmetrical with respect to line $r = a$, if $\bar{x}$ denote distance of centre of gravity of loop from origin, $\bar{x} = \frac{\frac{3}{2} \int r^3 \cos \theta d\theta}{\int r^2 d\theta}$, the integrations

being taken between $\theta = \frac{\pi}{2n}$ and $\theta = -\frac{\pi}{2n}$. But

$$\begin{aligned} \int r^3 \cos \theta \, d\theta &= a^3 \int \cos^3 n\theta \cos \theta \, d\theta = \frac{1}{4}a^3 \int (\cos 3n\theta + 3 \cos n\theta) \cos \theta \, d\theta \\ &= \frac{1}{4}a^3 \int [\cos (3n+1)\theta + \cos (3n-1)\theta + 3 \cos (n+1)\theta + 3 \cos (n-1)\theta] \, d\theta \\ &= \frac{1}{4}a^3 \left[\frac{\sin (3n+1)\theta}{3n+1} + \frac{\sin (3n-1)\theta}{3n-1} + \frac{3 \sin (n+1)\theta}{n+1} + \frac{3 \sin (n-1)\theta}{n-1} \right]_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} \\ &= \frac{1}{4}a^3 \cdot \frac{8n^2}{(n^2-1)[(3n)^2-1]} \cdot 6n \sin \left(\frac{n+1}{2n} \right) \pi, \text{ between the limits.} \end{aligned}$$

$$\text{Again, } \int r^2 \, d\theta = a^2 \int \cos^2 n\theta \, d\theta = (\text{between same limits}) \frac{a^2 \cdot \pi}{2n},$$

$$\therefore \bar{x} = \frac{\frac{1}{4} \cdot \frac{1}{4}a^4 \cdot \frac{8n^2}{(n^2-1)[(3n)^2-1]} \cdot 6n \sin \left(\frac{n+1}{2n} \right) \pi}{\frac{a^2 \pi}{2n}} = \frac{16an^4 \sin \left(\frac{n+1}{2n} \right) \pi}{(n^2-1)[(3n)^2-1]\pi}.$$

7181. (By H. L. ORCHARD, M.A.)—Given that the sum of the radii of the inscribed, escribed, and circumscribed circles of a certain equilateral triangle is unity, find the area of the triangle.

Solution by C. MORGAN, M.A.; R. KNOWLES, B.A.; and others.

If x = side of equilateral triangle, we have

$$R = \frac{1}{3}\sqrt{3} \cdot x, \quad r = \frac{1}{3}\sqrt{3} \cdot x, \quad r_a + r_b + r_c = \frac{3}{2}\sqrt{3} \cdot x;$$

and, equating the sum of these to unity, we get

$$x = \frac{1}{3}\sqrt{3}, \quad \text{and} \quad \text{area} = \frac{1}{4}\sqrt{3}.$$

7175. (By G. G. MORRICE, B.A.)—Prove that the angle between any two straight lines in space $= \sin^{-1} |l_1, m_2, n_3|$, where l_1, m_1, n_1 , l_2, m_2, n_2 are their direction-cosines, and l_3, m_3, n_3 those of their shortest distance.

Solution by G. TURRIFF, M.A.; W. M. MEE, B.A.; and others.

$$\frac{l_3}{m_1 n_2 - m_2 n_1} = \frac{m_3}{n_1 l_2 - n_2 l_1} = \frac{n_3}{l_1 m_2 - l_2 m_1} = \pm \frac{1}{\sin \theta}$$

(FROST'S *Solid Geometry*, Art. 23),

therefore $\sin \theta = \pm [l_3 (m_1 n_2 - m_2 n_1) + m_3 (n_1 l_2 - n_2 l_1) + n_3 (l_1 m_2 - l_2 m_1)]$

$$= \pm \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

7171. (By E. W. SYMONS, M.A.)—Prove that, in any triangle,
 $\cot^2 A + \cot^2 B + \cot^2 C > 1$.

Solution by H. L. ORCHARD, M.A.; J. S. JENKINS, M.A.; and others.

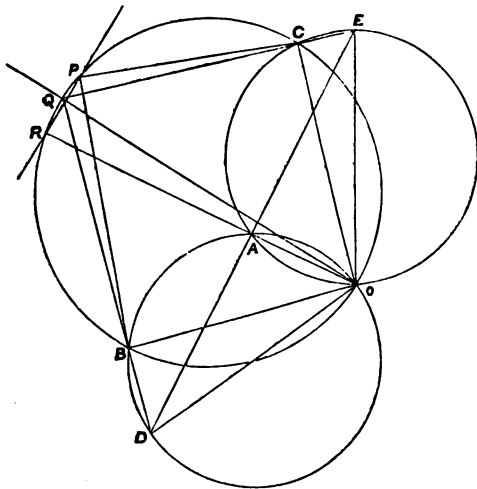
$$\begin{aligned} \sin^2 A \sin^2 B \cos^2 C + \sin^2 B \sin^2 C \cos^2 A + \sin^2 C \sin^2 A \cos^2 B &> 0, \\ \therefore, \quad \sin^2 A \sin^2 B \cos^2 C + \sin^2 B \sin^2 C \cos^2 A &+ \sin^2 C \sin^2 A (\cos^2 B - \sin^2 B) > 0, \\ \sin^2 A \sin^2 B \cos^2 C + \sin^2 B \sin^2 C \cos^2 A &+ \sin^2 C \sin^2 A \cos^2 B > \sin^2 A \sin^2 B \sin^2 C, \\ \text{and hence} \quad \cot^2 A + \cot^2 B + \cot^2 C &> 1. \end{aligned}$$

7112. (By R. F. SCOTT, M.A.)—Three circles which pass through the same point O, intersect again in A, B, C. Through the point A a line is drawn meeting the circles which intersect at A in D and E. The straight lines DB, EC meet the third circle again in F and G. Prove that, as the line DAE varies, the straight line FG envelopes a circle concentric with the circle BOC.

Solution by J. W. SHARPE, M.A.; E. RUTTER; and others.

Let the lines DB and EC meet at P; then

$$\angle PDA \text{ or } \angle BDA = \angle AOB, \text{ also } \angle PEA \text{ or } \angle CEA = \angle AOC,$$



therefore $\angle PDA + \angle PEA = \angle BOC$;

therefore $\angle BPC$ and $\angle BOC$ are supplementary; and therefore P lies on the third circle. Hence the points F and G are coincident.

The same result may be otherwise obtained analytically, and in the course of the investigation it may easily be shown that the lines DBP, ECP are equally inclined to BQ, CQ respectively, where OQ is a diameter of the third circle, and that the angles they make with these lines are complementary to that which DAE makes with OA, as is geometrically evident; furthermore, that PR is parallel to DAE, which also is easily proved geometrically. Thus we have the following theorem:—If the radical axis OA of any two of the circles be produced to cut the third circle in the point R, and the line RP be thus determined, then the three lines such as RP form a triangle whose sides are parallel respectively to those of PDE.

7115. (By A. McMURCHY, B.A.)—If $a_1, a_2, a_3 \dots$ be the lengths of the sides of a convex polygon ABCD... inscribed in a circle; $p_1, p_2, p_3 \dots$ the lengths of the perpendiculars on the considered position from any point P on the smaller arc cut off by a_1 ; prove that

$$\frac{a_1}{p_1} = \frac{a_2}{p_2} + \frac{a_3}{p_3} + \dots + \frac{a_n}{p_n}.$$

Solution by J. O'REGAN; J. W. SHARPE, M.A.; and others.

Let ABC be a triangle inscribed in a circle, P being on the arc AB; then $\Delta PP_2P_1 + \Delta PP_1P_3 = \Delta PP_2P_3$, i.e., $p_1 p_2 \sin B + p_1 p_3 \sin A = p_2 p_3 \sin C$; but $\frac{\sin B}{a_3} = \frac{\sin A}{a_2} = \frac{\sin C}{a_1}$, therefore $\frac{a_1}{p_1} = \frac{a_2}{p_2} + \frac{a_3}{p_3}$.

Now, in case of a polygon, draw the line AC, and let q be the perpendicular thereon; then

$$\frac{a_1}{p_1} = \frac{a_2}{p_2} \frac{b_1}{q_1}, \quad \frac{b_1}{q_1} = \frac{a_3}{p_3} + \frac{b_2}{q_2}, \dots, \quad \frac{b_{n-3}}{q_{n-3}} = \frac{a_{n-1}}{p_{n-1}} + \frac{a_n}{p_n}; \quad \therefore \frac{a_1}{p_1} = \frac{a_2}{p_2} + \dots + \frac{a_n}{p_n}.$$

7213. (By C. MORGAN, B.A.)—Show that, in any triangle,

$$s \cdot \Sigma (r_a^{-1}) = \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C,$$

where (as usual) r_a is the radius of the circle escribed to the side a , and s is the semiperimeter.

Solution by R. KNOWLES, B.A.; G. TURRIFF, M.A.; and others.

The first member = $\Delta^{-1} s (s_1 + s_2 + s_3)$, where $s_1 = s - a$, &c. = $\Delta^{-1} s^2$; the second = $s^{\frac{1}{2}} (s_1 s_2 s_3)^{-\frac{1}{2}} = \Delta^{-1} s^2$.

5689. (By Professor CAYLEY, F.R.S.)—Show (1) that the apparent contour of a Steiner's surface ($2xyz + y^2z^2 + z^2x^2 + x^2y^2 = 0$), as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $(\pm 1, 0)$ and $(0, \pm 1)$; and (2) find the surface-contour, or curve of contact of the cone and surface.

Solution by W. J. C. SHARP, M.A.; Professor MATZ, M.A.; and others.

The equation to the tangent plane to the Steiner's surface at (x, y, z) may be put in the form $\frac{yz}{x}(yz+x)\xi + \frac{zx}{y}(zx+y)\eta + \frac{xy}{z}(xy+z)\zeta = xyz$, where ξ, η, ζ are the current coordinates.

If this plane cut the nodal line (axis of z) at a distance a from the origin, $(xy+z)a = x^2$, and from this equation combined with $y^2z^2 + z^2x^2 + x^2y^2 = 0$ it follows that $x^2 + y^2 + \frac{(z-a)^2}{a^2} + 2\frac{z-a}{a} = 0$, that is, $x^2 + y^2 + \frac{z^2}{a^2} - 1 = 0$, a quadric of revolution on which the curve of contact lies.

The equations to the line joining $(0, 0, a)$ to (x, y, z) , the point of contact, are $\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta-a}{z-a}$, and this is the generator of the enveloping cone; at the points where this meets the plane of $xy, \zeta = 0$,

and therefore $x = -\frac{\xi(a-z)}{a}, y = -\frac{\eta(z-a)}{a}$;

and therefore $xy = \frac{\xi\eta}{a^2}(z-a)^2 = \frac{z(z-a)}{a}$, from above;

and therefore $\frac{z-a}{z} = \frac{a}{\xi\eta}$; and therefore $\frac{z-a}{a} = \frac{a}{\xi\eta-a}$;

$\therefore$ since $x^2 + y^2 + \frac{(z-a)^2}{a^2} + 2\frac{z-a}{a} = 0$, $(\xi^2 + \eta^2 + 1)\frac{a^2}{(\xi\eta-a)^2} + 2\frac{a}{\xi\eta-a} = 0$.

Therefore $(\xi^2 + \eta^2)a + 2\xi\eta - a = 0$ is the equation to the apparent contour an ellipse through the four points.

6979. (By Professor TOWNSEND, F.R.S.)—A spiral intersecting at any constant angle all the meridians of any convex surface of revolution, being supposed the frictionless catenary of a uniform cord, or the frictionless trajectory of a material particle, constrained to rest or move on the surface, under the action of a force passing perpendicularly in every position through the axis of revolution; show that the force varies, for the catenary inversely as the square, and for the trajectory inversely as the cube, of the distance from the axis.

Solution by the PROPOSER.

Denoting by ρ and ψ the axial distance and azimuthal angle of any point on the spiral, by T and v the tension of the cord and the velocity of the

particle at the point, and by k the cosecant of the constant angle of intersection of the curve with all the meridians of the surface; then since, on elementary dynamical principles, $dT = -Fdp$, and $T\rho^2 d\psi = hds$, where h is a constant, for the catenary, and, $v dv = Fdp$, and $v\rho^2 d\psi = hds$, where h again is a constant, for the trajectory, and since, from the fundamental property of the spiral, $ds = k\rho d\psi$ for both cases alike, therefore at once, $F = hk \cdot \rho^{-2}$ for the catenary, and $= -h^2 k^2 \cdot \rho^{-3}$ for the trajectory; and therefore, &c.

7033. (By Professor HUDSON, M.A.)—A particle is attracted to one and repelled from another centre of force varying inversely as the square of the distance and of equal absolute intensity; show that the surfaces of equilibrium are those formed by the revolution of parts of the curves

$$(r^2 + a^2)^3 (r^2 + a^2 - 2c^2) + 4a^2 r^2 \cos^2 \theta [c^4 - 2(r^2 + a^2)(r^2 + a^2 - c^2)], \\ + 16a^4 r^4 \cos^4 \theta = 0$$

where $2a$ is the distance between the centres of force. What do the other parts of these curves represent?

Solution by W. H. BLYTHE, M.A.; J. HAMMOND, M.A.; and others.

If r_1, r_2 be the distances of the particle from the centres of force respectively, the surfaces of equilibrium are given by

$$\int \left(\frac{\mu dr_1}{r_1^3} - \frac{\mu dr_2}{r_2^3} \right) = \text{constant} \dots \dots \dots (1)$$

Let $2a$ be the distance between the centres, take the middle point as origin polar coordinates, then

$$r_1^2 = r^2 + a^2 - 2ar \cos \theta \equiv t - p \text{ (suppose),} \\ r_2^2 = r^2 + a^2 + 2ar \cos \theta \equiv t + p \text{ (suppose).}$$

Integrate (1) and square; then, the constant being arbitrary, we can take as solution $c^2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right)^2 = 2$; and, substituting the values for r_1 and r_2 ,

$$c^2 [(t+p) - 2(t+p)^{\frac{1}{2}}(t-p)^{\frac{1}{2}} + (t-p)] = 2(t^2 - p^2),$$

or

$$c^2(t^2 - p^2)^{\frac{1}{2}} = t^2 - p^2 - c^2 t.$$

Squaring, $c^4(t^2 - p^2) = t^4 + p^4 + c^4 t^2 - 2t^2 p^2 - 2c^2 t^3 + 2c^2 p^2 t \dots \dots \dots (2)$;

therefore $t^3(t - 2c^2) + p^3[c^4 - 2t(t - c^2)] + p^4 = 0$, or

$$(r^2 + a^2)^3 (r^2 + a^2 - 2c^2) + 4a^2 r^2 \cos^2 \theta [c^4 - 2(r^2 + a^2)(r^2 + a^2 - c^2)] + 16a^4 r^4 \cos^4 \theta.$$

It is to be noticed that equation (2) is obtained also by simplifying $c^2 \left(\frac{1}{r_1} + \frac{1}{r_2} \right)^2 = 2$, and thus includes also the surfaces of equilibrium when both forces are attractive or both repulsive.

4971. (By Professor SYLVESTER, F.R.S.)—The equation to the inverse of a conic in Cartesian coordinates being given, find the algebraical expression for the node in its simplest terms.

Solution by W. J. C. SHARP, M.A.; ELIZABETH BLACKWOOD; and others.

If $ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0$ be the conic referred to any rectangular axes, and m the radius of inversion, the inverse is

$$m^4(aX^2 + 2hXY + bY^2) + 2(fY + gX)m^2(X^2 + Y^2) + c(X^2 + Y^2)^2 = 0,$$

and therefore the nodal tangents at the origin are parallel to the asymptotes to the conic, and the node is a crunode, a cusp, or an acnode, according as $h^2 - ab$ is positive, zero, or negative. More generally, the inverse of an π -ic, which does not pass through the circular points, has an π -ic point at the origin, the tangents at which are parallel to the asymptotes of the curve inverted.

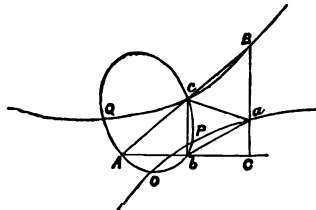
7075. (By Professor WOLSTENHOLME, M.A.)—If in a triangle ABC there be inscribed another triangle abc , and two conics through Bca , Cab , intersect again in A' , B' , C' ; prove that the conic inscribed in ABC and touching two sides ($A'B'$, $A'C'$) of the triangle $A'B'C'$ will have double contact with the conic through the five points a , b , c , B' , C' .

Solution by the PROPOSER.

Given a triangle ABC and a conic U , meeting the sides of the triangle in the points a, a' ; b, b' ; c, c' respectively; then four conics can be inscribed in the triangle which shall have double contact with the conic. One of them is determined by the following construction: draw a conic V_1 through Abc meeting U again in P, Q , the conic V_2 through Bca PQ (as also the conic V_3 through $Ca bPQ$) will meet V_1 in a fourth point O , and the (unique) conic inscribed in ABC and touching OP, OQ will have double contact with U . The same conic might also be determined by taking $a'b'c'$ in place of abc ; and the other three will be found in the same way from the groups of points $a'bc$ or $ab'c'$; $ab'e$ or $a'b'd'$; abd' or $a'b'c$. These four conics will of course always be real when the points abc $a'b'c'$ are real, and the chord of contact will then also be real, but the points of contact may be impossible.

The property, assumed in the construction, that the conics V_1, V_2, V_3 have a common point O (the projection of a well known property of circles), is readily proved from the fundamental anharmonic point property as follows: Let abc be a triangle inscribed in ABC , V_1 any conic through Abc , V_2 a conic through Bca meeting V_1 also in OPQ , then if we consider the six points Cab OPQ , we have

$$\begin{aligned} a | COPQ | &= a | BOPQ | = c | BOPQ | \\ &= c | AOPQ | = b [AOPQ] = b [COPQ]; \end{aligned}$$



hence $b \mid \text{COPQ} \mid = a \mid \text{COPQ} \mid$, or the six points abC OPQ lie on one conic.

This is of course equivalent to the theorem that the circles Abe , Bca , Cab , concur in a point O . Then I further assert that the (unique) conic touching OP , OQ and the sides of the triangle ABC will have double contact with the conic through abc PQ .

6932. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—It is an old theorem that every $(12+r)$ -edron P , whose summits are all triedral, and whose faces are all 5-gons and 6-gons, has twelve pentagons. Required the number of these solids (P) for values of r up to $r=7$, no one being counted which can be drawn in any position either as the repetition or the reflected image of another; together with a complete account of symmetry, enumerating the symmetrical faces, summits, and edges. The required number is less than 20.

Solution by the PROPOSER.

The solids required are—for $r=0$, one zoned hexarchaxine, whose six principal axes have 5-zoned 5-gons, whose ten secondary axes have 3-zoned triaxes, and whose fifteen tertiary axes have 2-zoned edges (55), for poles. The 31 axes are all janal.

For $r=2$, one 6-zoned monarchaxine homozone, whose zoned principal axis has 6-gons, and whose six zoneless secondary axes have twelve edges (55), for zoneless poles. All axes janal.

For $r=3$, A_3 , a 3-zoned monarchaxine heterozone, has 3-zoned triaxes for poles of the principal axis, while its three like secondary heteroid (=non-janal) axes have each for 2-zoned poles a 6-gon and an edge 55. (A monarchaxine homozone has only one zoned axis and not two different zones; a monarchaxine heterozone has more than one zoned axis and has different zones.)

For $r=4$, A_4 , a zoned tetrarchaxine, has for poles terminating each of its four principal 3-zoned heteroid axes, a 3-zoned 6-gon and triaxe; while its three secondary janal 2-zoned axes carry one zoned polar edge (55). Also B_4 is a zoneless triaxine, whose three different 2-ple janal axes have, one poles (66), a second poles (55), and the third other poles (55), all zoneless.

For $r=5$; A_5 is a 2-zoned monaxine heteroid, whose poles are a 2-zoned 6-gon and an edge (55). B_5 is another 2-zoned monaxine heteroid, whose axis carries a zoned polar edge (66) and a 2-zoned 6-gon. A third solid is C_5 , a 5-zoned monarchaxine heterozone, on which a 5-gon terminates the principal janal axis, while the five secondary heteroid 2-zoned axes have each for poles a 6-gon and an edge (66).

For $r=6$; A_6 is a 3-zoned monarchaxine heterozone: the poles of the janal principal axis are 3-zoned triaxes, and the three 2-zoned heteroid secondaries have for poles each two different edges (55). Next B_6 is a 3-ple zoneless monarchaxine. The two principal janal poles are the same 3-ple triaxe, and the three heteroid secondary axes have each for poles (66) and (55). A third solid is C_6 , a 2-ple zoneless monaxine heteroid, whose poles are two different (55). Similar to C_6 is D_6 , whose 2-ple poles are (66) and (55). Next E_6 is a 3-zoned monarchaxine homozone, whose

principal janal poles are 3-zoned triaces, its six janal secondary poles of three zoneless axes being the same zoneless polar (66). A sixth solid is F_6 , a zoneless triaxine, whose six janal 2-ple poles are a 6-gon, an edge (66) and an edge (55), on three different janal axes.

For $r = 7$; A_7 is a 3-zoned monaxine heteroid on whose axis the 3-zoned poles are a triace and a 6-gon. B_7 is a 2-ple zoneless monaxine heteroid whose poles are a 2-ple 6-gon and an edge (55). C_7 is a monone, whose zoned faces are four 5-gons, and one 6-gon with diagonal trace. It has three zonal and two epizonal edges. D_7 is a monozone, having 2 zoned 5-gons, one 6-gon with agonal trace and two 6-gons diagonally traced. It has also three zonal and two epizonal edges. E_7 and F_7 are 2-ple monaxine heteroids, of which the first has for zoneless poles a 2-ple 6-gon and edge (66) and the second, like B_7 , a 6-gon and (55).

These nineteen, one 14-edron, one 15-edron, two 16-edra, three 17-edra, six 18-edra, and six 19-edra, are the solids required for $r > 0$, up to $r = 7$.

It would be unbecoming not to remark, before I give my way (it will come better with the solution of Question 6964) of finding and construction on those for lower values of these results, that Professor Cayley has invented (Memoirs of Literary and Philosophical Society of Manchester, Third Series, Vol. I., 1862, p. 248) a method of constructing from triangular-faced *data*, the P-acral Q-edra whose faces are all triangles; or, what is the same thing, the P-edral Q-acra whose summits are all triaces, on data with like summits. The most important thing in his paper is a special method of solving the questions before us here, and any extension of them.

This remark would be called for, if the writer were even far below the stature of a Cayley, on the only attempt in any language that I have heard of, besides my own humble efforts, to touch the Polyedra.

Professor Cayley has made of this special method no application whatever, of which I am aware, neither have I. The only results given in his paper are the enumeration and construction of the 8-acra whose faces are triangular, with triaces and tesseraces among the summits, that is of the 8-edra whose summits are tri-edral, without exclusion of triangles and 4-gons. This he does evidently unaware at the time that I had enumerated and constructed the same solids in the same Memoirs, Vol. XII., 1854. My triplets are neither more nor less than his triangles. But one defect runs through both Professor Cayley's paper and my earlier one, in that no correct account is given of symmetry. Nothing would give me more pleasure, in connexion with Polyedra, than to see Professor Cayley's method applied to obtain the results above given, together with a few more as far as $r = 10$ or $r = 12$. I am persuaded that, backed by sufficient industry and very skilful ordering of data and results, the method will do the work, if only the worker be master of the grand puzzle of polyedral symmetry. For information on that head, I can only refer to a chapter, No. XXV., of the Philosophical Transactions for 1862, in which will be found definitions and explanations of the terms above employed. There is much to be discovered in choosing out of a number the most elegant method of constructing detached sub-families of polyedra, so as to preclude omissions and repetitions. In the general theory, the enumeration of the triangular-faced solids from those of one edge fewer is, beyond all comparison, the simplest and briefest task in this study that can be set. When the enumeration and description are completed, the construction of the solids is a bagatelle, with the tables of edges all before us. For an example, see the "Proceedings of the Literary and Philosophical Society of Liverpool," Vol. XXXII.; 1877-8.

1100. (By the EDITOR.)—If a straight line be divided at random into three parts, show that the chance (1) that a triangle of any kind can be formed with the three parts is $\frac{1}{4}$; and (2) that an acute-angled triangle can be formed by the three parts is $3 \log 2 - 2$.

Solution by W. T. A. EMTAGE; Prof. MATZ, M.A.; and others.

1. Let $x, y - x, 2s - y$ be the three parts; then the required chance is

$$\iint dy dx + \iint dy dx,$$

where the limits in the numerator are given by $x < s, y > s, y - x < s$; and in the denominator x and y have all values possible, y being $> x$.

Thus the required chance becomes

$$\int_s^{2s} \int_{y-s}^s dy dx + \int_0^{2s} \int_0^y dy dx = \int_s^{2s} (2s - y) dy + \int_s^{2s} y dy = \frac{2s^2 - \frac{1}{2}(4s^2 - s^2)}{\frac{3}{2}s^2} = \frac{1}{4}.$$

2. To find the chance that a triangle is formed, and that the segment x subtends an obtuse angle, we have the condition

$$x^2 > (y - x)^2 + (2s - y)^2, \therefore 0 > 2y^2 - 2xy + 4s^2 - 4sy, \therefore x > y + \frac{2s^2}{y} - 2s;$$

hence the numerator of the fraction giving the chance is

$$\begin{aligned} \int_s^{2s} \int_{y-2s+\frac{2s^2}{y}}^s dy dx &= \int \left(3s - y - \frac{2s^2}{y} \right) dy \\ &= 3s^2 - \frac{3}{2}s^2 - 2s^2 \log 2 = \frac{3}{2}s^2 - 2s^2 \log 2, \end{aligned}$$

and the denominator is the same as before; consequently, the required chance is $\frac{3}{2} - \log 2$.

So the chance that either of other segments subtends an obtuse angle is $\frac{3}{2} - \log 2$; and as the three cases in which an obtuse-angled triangle is made are mutually exclusive, the chance that an obtuse-angled triangle is made at all is $\frac{3}{2} - 3 \log 2$; and the chance that a triangle is made being $\frac{1}{4}$, the chance that an acute-angled triangle is made is

$$\frac{1}{4} - \left(\frac{3}{2} - 3 \log 2 \right) = 3 \log 2 - 2.$$

[For other solutions, see *Reprints*, II., 74, 75; V., 33; XV., 35, 36.]

6966. (By the Rev. W. A. WHITWORTH, M.A.)—Show that there are 220 triangles whose sides are integers not exceeding 100, and whose areas are also integral.

Solution by G. HEPPLE, M.A.; B. REYNOLDS, M.A.; and others.

It is impossible, without going to a great length, to do more than give a sketch of the method of determining the triangles. It is easy to see that the sines and cosines of all the angles must be rational fractions. Using the general form $\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2}$, make a list of rational sines and cosines with denominators not greater than 100,

rejecting all that are not in their lowest terms. There will be 16 of these, which will give us at once as many right-angled triangles. Of some of these multiples may be taken, for example, multiples up to 20 of 3, 4, 5; multiples up to 7 of 5, 12, 13, and so on. In this way there will be found to be 52 right-angled triangles.

Next, to every such right-angled triangle with sides abc , (c being hypotenuse) there must be two corresponding isosceles triangles $c, c, 2a$, and $c, c, 2b$. Of these there will be 23 with sides not greater than 100, and taking multiples as before, the number will be increased to 72.

Lastly, combine any other pairs of the original list of sines and cosines, determine a third fraction by the $\sin(A+B)$ formula, and take the least three integers proportional to the three fractions. Reject those containing numbers over 100, and those which are repetitions. This will furnish 56 scalene triangles, and with the multiples 96. Hence the total number is, as stated in the Question, 220.

The list of the 220 triangles is as set forth hereunder:—

(1) Right-angled; in number including multiples, 62.

3	4	5	20	21	29	11	60	61	36	77	85
5	12	13	12	35	37	16	63	65	13	84	85
8	15	17	9	40	41	33	56	65	39	80	89
7	24	25	28	45	53	48	55	73	66	72	97

(2) Isosceles; in number, including multiples, 72.

5	5	6	25	25	14	41	41	18	65	65	66
5	5	8	25	25	48	41	41	80	73	73	96
13	13	10	29	29	40	53	53	66	85	85	72
13	13	24	29	29	42	53	53	90	85	85	26
17	17	16	37	37	24	61	61	22	89	89	78
17	17	30	37	37	70	65	65	32			

(3) Scalene; in number, including multiples, 96.

4	13	15	25	29	36	34	65	93	53	75	88
13	14	15	25	39	40	35	53	66	57	82	89
7	15	20	29	35	48	36	61	65	56	61	75
11	13	20	39	41	50	37	91	96	57	65	68
10	17	21	13	68	75	39	41	50	60	73	91
12	17	25	15	41	52	39	85	92	61	74	87
13	20	21	17	55	60	40	51	77	61	91	100
17	25	26	20	37	51	41	51	58	65	76	87
17	25	28	25	39	56	41	84	85	65	87	88
13	37	40	25	52	63	43	61	68	68	75	77
13	40	45	25	51	52	48	85	91	68	87	96
15	34	35	25	74	77	50	69	73	75	86	97
15	37	44	26	51	55	51	52	53	78	95	97
17	39	44	29	52	69	52	73	75	89	99	100.

6872. (By R. KNOWLES, B.A., L.C.P.)—In a central conic, if GK is the perpendicular from G, the foot of the normal at P on CP drawn from the centre C, prove that the rectangle CP . PK is constant.

Solution by A. COHEN, Q.C., M.P.; SARAH MARKS; and others.

Draw CU perpendicular on the tangent. Then triangles CUP, GPK are similar. Therefore $PK \cdot PC = PG \cdot CU$. But the product of the normal and the perpendicular on the tangent equals b^2 . Therefore $PK \cdot PC = b^2$.

7001. (By T. WOODCOCK, B.A.)—The straight line through P, any point on the circumscribing circle of a triangle ABC, drawn perpendicular to the pedal line of P, meets the sides of the triangle in x, y, z : and the straight line through the orthocentre, parallel to the pedal line of P, meets the sides in X, Y, Z; Pp, Pq, Pr are drawn at right angles to PX, PY, PZ to meet the corresponding sides in p, q, r ; show that (1) xp, yq, zr are bisected by the pedal line of P; (2) the triangle pqr is double of the triangle ABC; (3) PAq, PrA; PBr, PpB; and PCp, PqC are pairs of similar triangles.

Solution by the PROPOSER; E. RUTTER; and others.

The lines XYZ, xyz , and the pedal line of P will be the directrix, axis, and tangent at the vertex respectively, of a parabola with focus P and touching the sides of the triangle in p, q, r : and the above are well known properties of this parabola.

6809. (By J. L. MCKENZIE, B.A.)—(1) Let O be the orthocentre of a triangle ABC, and N the centre of the nine-points circle. Let the circle through O, and through the middle points of OB and OC, cut the line NO in P; and let the two other analogous circles cut NO in Q and R. Prove that the lines joining P, Q, and R with the middle points of BC, BA, and AB, all intersect in a point S on the circumference of the nine-points circle. (2) Let the circle through N, and through the middle points of OB and OC, cut NO in P', and similarly for two other circles; prove that the lines joining P', Q', R' to the feet of the perpendiculars on BC, CA, and AB from the opposite angles, all intersect in the same point S as before.

Solution by the PROPOSER; PROFESSOR MATZ, M.A.; and others.

It is easily seen, on drawing a figure, that each of the three circles in the first part of the question is equal to the nine-point circle; and each circle in (2) has a diameter equal to half the diameter of a circle that passes through two angular points of the triangle ABC, and through the centre of the circumscribing circle. It can be proved, that if T be the other extremity of the chord of the nine-points circle drawn through S, the points T, N, O, and the middle points of OA, OB, and OC, are six points on the same hyperbola.

6790. (By G. F. WALKER, M.A.)—Three particles are at the corners of an equilateral triangle of side a , and are each projected parallel to the opposite side towards the same parts with a velocity $[af(a)]^{\frac{1}{2}}$, where $f(r)$ is the law of attraction between the particles; show that they will always be at the corners of an equilateral triangle.

Solution by the PROPOSER; Professor SEITZ, M.A.; and others.

Suppose a smooth wire ring to be drawn through the particles and that they are constrained to slide on it. Let R be the pressure. The particles being all set off in the same direction round the circle, and with the same velocity, it is clear that their motion will continue unaltered. We have then $m \frac{af(a)}{a} = R + \frac{1}{2} \cdot 2f(a) \cdot m$, therefore $R = 0$; and the wire may be removed without altering the motion.

6899. (By R. TUCKER, M.A.)—Show that, in the curve whose equation is $b^2e^{\frac{x}{b}} + a^2e^{-\frac{x}{a}} = 2a^2 \cos \frac{a}{b}$, there are pairs of radii of curvature (ρ, ρ') , so related that $\rho^2 + \rho'^2 = a^2 + b^2$.

Solution by the PROPOSER.

Referring to Vol. I. of the *Reprint*, it will be seen that the radial of the curve in question is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, of which it is a well-known property that $CP^2 + CD^2 = a^2 + b^2$; whence, &c.

6895. (By Professor MATZ, M.A.)—Find the value of ϕ in the equation $\pi \tan^2 \phi + \sqrt{(2)} - \sqrt{(3)} = -3 [\pi \sqrt{(6)} \tan^2 \phi]^{\frac{1}{2}}$.

Solution by W. H. BLYTHE, B.A.; H. L. ORCHARD, M.A.; and others.

The equation is of the form $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$; where $a^2 = \pi \tan^2 \phi$, $b^2 = \sqrt{2}$, $c^2 = -\sqrt{3}$. Since b and c are real and unequal, the roots of the quadratic factor are imaginary; taking, therefore, $a+b+c=0$, and solving for $\tan \phi$, we find $42' 38'' \cdot 5$ as the least value for the angle ϕ , all values included in $n\pi \pm a$.

6944. (By W. GALLATLY, B.A.)—Prove that, in any triangle ABC,

$$\frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} + \frac{\sin \frac{1}{2}(C-A)}{\sin \frac{1}{2}(C+A)} + \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \\ + \frac{\sin \frac{1}{2}(B-C) \cdot \sin \frac{1}{2}(C-A) \cdot \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(B+C) \cdot \sin \frac{1}{2}(C+A) \cdot \sin \frac{1}{2}(A+B)} = 0.$$

Solution by A. McMURCHY, B.A.; J. O'REGAN; and others.

The first term is $\frac{\sin B - \sin C}{\sin A} = \frac{b-c}{a}$, and, treating all the terms similarly, we have

$$\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} + \frac{(b-c)(c-a)(a-b)}{abc},$$

which vanishes identically.

6756. (By E. B. ELLIOTT, M.A.)—If u be a homogeneous function of the n^{th} degree in x_1, x_2, x_3, x_4 , four variables u_1, u_2, u_3, u_4 its first, and $u_{11}, u_{12}, \dots, u_{44}$ its second partial differential coefficients, prove (1) that

$$\begin{vmatrix} u_{11} & \dots & u_{14} \\ \dots & \dots & \dots \\ u_{41} & \dots & u_{44} \end{vmatrix} = \frac{(n-1)^2}{x_4^2} \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_1 \\ u_{21} & u_{22} & u_{23} & u_2 \\ u_{31} & u_{32} & u_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} + \frac{n(n-1)}{x_4^2} u \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix};$$

and (2) that corresponding relations hold for any number of variables.

Solution by W. J. C. SHARP, M.A.; Professor NASH, M.A.; and others.

The value of the Hessian given in the Question is an expansion of the expression

$$\frac{n-1}{x_4^2} \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_1 \\ u_{12} & u_{22} & u_{23} & u_2 \\ u_{13} & u_{23} & u_{33} & u_3 \\ (n-1)u_1 & (n-1)u_2 & (n-1)u_3 & nu_4 \end{vmatrix},$$

which I gave in a note to a solution of Quest. 5762 (*Reprint*, Vol. xxxiii., p. 92). The same method gives the generalized result

$$\frac{(n-1)^2}{x_n^2} \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} & u_1 \\ \dots & \dots & \dots & \dots & \dots \\ u_{1,n-1} & u_{2,n-1} & \dots & u_{n-1,n-1} & u_{n-1} \\ u_1 & u_2 & \dots & u_{n-1} & 0 \end{vmatrix} \\ + \frac{n(n-1)}{x_n^2} \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} \\ u_{12} & u_{22} & \dots & u_{2,n-1} \\ \dots & \dots & \dots & \dots \\ u_{1,n-1} & u_{2,n-1} & \dots & u_{n-1,n-1} \end{vmatrix}.$$

These forms furnish a direct proof that, no matter how many the variables, the Hessian is satisfied by all the singular roots (SALMON'S *Higher Algebra*, p. 85) of the quantic.

6764. (By the Rev. T. R. TERRY, M.A., F.R.A.S.)—Show that, unless m is a negative integer, the limit, when p is indefinitely increased, of

$$\frac{m(m+1)(m+2)\dots(m+2p-1)}{1.3.5\dots(2p-1)2m(2m+2)\dots(2m+2p-2)} \text{ is } 2^{m-1}.$$

Solution by E. HAIGH, B.Sc., B.A.; BELLE EASTON; and others.

$$\text{Let } u = \frac{m(m+1)(m+2)\dots(m+2p-1)}{1.3.5\dots(2p-1)2m(2m+2)\dots(2m+2p-2)},$$

$$\text{then } u = \frac{m+2p-1}{m+p-1} \cdot \frac{p!}{2p!} \dots\dots\dots (1),$$

m being assumed to be a positive integer.

By a known theorem, when x is infinite, the limit of $1.2\dots x$ is

$$(2\pi x)^{\frac{1}{2}} x^x e^{-x} \text{ or } Kx^{x+\frac{1}{2}} e^{-x},$$

where K is a constant. Hence, in the limit, when p is infinite,

$$u = \frac{(m+2p-1)^{m+2p-\frac{1}{2}} p^{p+\frac{1}{2}}}{(m+p-1)^{m+p-\frac{1}{2}} (2p)^{2p+\frac{1}{2}}} \dots\dots\dots (2),$$

e disappearing from the ratio by virtue of the relation

$$(m+2p-1) + p = (m+p-1) + 2p.$$

Again, let x and z be two infinite quantities of the same order, whose difference is always finite and ratio is unity. Let $a = m-1$; then

$$\begin{aligned} (x+a)^z &= x^z \left(1 + \frac{a}{x}\right)^z = x^z \left\{1 + z \frac{a}{x} + \frac{z(z-1)}{1.2} \frac{a^2}{x^2} + \dots\right\} \\ &= x^z \left\{1 + \frac{z}{x} a + \frac{\frac{z}{x} \left(\frac{z}{x} - \frac{1}{x}\right)}{1.2} a^2 + \dots\right\} = x^z \left\{1 + a + \frac{a^2}{1.2} + \dots\right\} = x^z e^a. \end{aligned}$$

Putting $x = 2p$, $z = m+2p-\frac{1}{2}$, we obtain

$$(m+2p-1)^{m+2p-\frac{1}{2}} = e^{m-1} (2p)^{m+2p-\frac{1}{2}};$$

and, similarly, by substituting $x = p$, $z = m+p-\frac{1}{2}$,

$$(m+p-1)^{m+p-\frac{1}{2}} = e^{m-1} p^{m+p-\frac{1}{2}}.$$

Hence, substituting in (2),

$$u = \frac{(2p)^{m+2p-\frac{1}{2}} p^{p+\frac{1}{2}}}{p^{m+p-\frac{1}{2}} (2p)^{2p+\frac{1}{2}}} = 2^{m-1} \dots\dots\dots (3).$$

[Putting $a = \frac{1}{2}m$, $r = 2$ in the well known formula

$$\Gamma(a) \Gamma\left(a + \frac{1}{r}\right) \dots \Gamma\left(a + \frac{r-1}{r}\right) = \Gamma(ra) \cdot (2\pi)^{\frac{r-1}{2}} \cdot r^{\frac{1}{2}-ra},$$

we have

$$\frac{\Gamma\left(\frac{1}{2}\right) \Gamma(m)}{\Gamma\left(\frac{1}{2}m\right) \Gamma\left[\frac{1}{2}(m+1)\right]} = 2^{m-1}.$$

But $\Gamma(x)$ is the limit of $\frac{1.2.3\dots p}{x(x+1)\dots(x+p-1)} p^{x-1}$,

whence, substituting, we get the required result.]

6876. (By the Rev. W. A. WHITWORTH, M.A.)—Without altering the perimeter, it is required to lengthen very slightly one of the sides of an equilateral triangle and to shorten another, so that the radius of the inscribed circle may become commensurable with the sides. The increment and decrement, in order to be inappreciable to the eye, must not exceed $\frac{1}{700}$ th part of the length of the sides. Determine the ratio in which the sides may be altered.

Solution by G. M. REEVES, M.A.; CHRISTINE LADD, B.A.; and others.

Let $a+x$, a , $a-x$ be the sides of the slightly altered triangle; then

$$r = \frac{a}{2\sqrt{3}} \left(1 - \frac{4x^2}{a^2}\right)^{\frac{1}{2}}; \text{ hence, putting } \frac{2x}{a} = \frac{1}{y}, \text{ we have}$$

$$\frac{1}{2} \left(1 - \frac{1}{y^2}\right) = \text{perfect square} = x^2 \text{ (suppose); whence } y^2 - 3x^2 = 1,$$

where $v = yz$; and the ordinary method of solution gives

$$y = \frac{1}{2} [(2 + \sqrt{3})^n + (2 - \sqrt{3})^n];$$

and thence, for $n = 5$, we have $y = 362$; therefore $x = \frac{1}{724}a$, and the sides then become proportional to 723, 724, 725, and r becomes $\frac{209}{724}$ of the mean side.

6542 & 6628. (By Professor SEITZ, M.A.)—A plane is passed through three random points within a sphere; show (1) that the average area of the section is $\frac{1}{2}\pi r^2$, and (2) that the average volume of the smaller of the two segments into which the sphere is divided is $\frac{9}{1538}\pi r^3$.

Solution by Prof. CAVALLIN, M.A.; the PROPOSER; and others.

Let A, B, C be the three random points, DHKL the section made by the plane through A, B, C, M its centre, O the centre of the sphere, and OP a radius of the sphere, such that AB is parallel to the plane MOP.

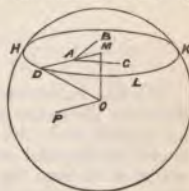
Let $OD = r$, $AM = x$, $AB = y$, $AC = z$, $MD = x'$, $u = \text{area DHKL}$, $v = \text{the volume of smaller segment}$, $\angle DOM = \theta$,

$$\angle MAB = \phi, \angle CAM = \psi, \angle MOP = \mu,$$

and the angle the plane MOP makes with a fixed plane through OP = ω .

Then $x' = r \sin \theta$, $u = \pi r^2 \sin^2 \theta$, $v = \frac{1}{3}\pi r^3 (2 - \sin^2 \theta \cos \theta)$; an element of the sphere at A is $r \sin \theta d\theta \cdot 2\pi x dx$, at B it is $y^2 dy d\phi d\mu$, and at C it is $\sin(\phi + \psi) \sin \mu z^2 dz d\psi d\omega$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x , 0 and x' , and tripled; of ϕ , $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ , $-\phi$ and $\frac{1}{2}\pi$, and doubled; of μ , 0 and π ; of ω , 0 and 2π ; of y , 0 and $2x \cos \phi = y'$; and of z , 0 and $2x \cos \phi = z'$. The whole number of ways the three points can be taken is $(\frac{3}{8}\pi r^3)^3$.



Put $\theta = FC$, $r\phi = PB$, $\lambda = GE$, $r\delta =$ a meridian through the summit and circumference of the base of the hill; then

$$d\theta = CD, \quad r d\phi = AE, \quad d\lambda = BE;$$

and

$$BA : CD = \sin PB : \sin PC;$$

or

$$BA : d\theta = \sin \phi : 1; \text{ whence } BA = \sin \phi d\theta.$$

The triangle BAE, being infinitely small, may be considered a plane triangle right-angled at A; therefore

$$(AE)^2 + (AB)^2 = (BE)^2, \text{ or } d\lambda^2 = r^2 d\phi^2 + \sin^2 \phi d\theta^2 \dots\dots\dots(1).$$

Raise a perpendicular from E; draw a horizontal line through B to intersect this perpendicular, and a right-angled triangle will be formed whose perpendicular is the differential of the perpendicular from the vertex of the hill on a plane through B parallel with the base of the hill, and whose base makes an angle β with AE. Therefore

$$r \sin \phi d\phi = AE \sin \beta, \text{ and } AE = d\lambda = \frac{r \sin \phi d\phi}{\sin \beta} \dots\dots\dots(2).$$

Equating the two values of $d\lambda$, we get

$$\begin{aligned} d\theta &= \frac{r d\phi (\sin^2 \phi - \sin^2 \beta)^{\frac{1}{2}}}{\sin \beta \sin \phi} = \frac{r (\sin^2 \phi - \sin^2 \beta) d\phi}{\sin \beta \sin \phi (\sin^2 \phi - \sin^2 \beta)^{\frac{1}{2}}} \\ &= \frac{r \sin \phi d\phi}{\sin \beta (\sin^2 \phi - \sin^2 \beta)^{\frac{1}{2}}} - \frac{r \sin \beta d\phi}{\sin \phi (\sin^2 \phi - \sin^2 \beta)^{\frac{1}{2}}}. \end{aligned}$$

Integrating, we obtain

$$\theta = r \operatorname{cosec} \beta \cos^{-1} (\sec \beta \cos \phi) - r \sin \beta \cos^{-1} (\tan \beta \cot \phi) + C.$$

When $\theta = 0$, $\phi = \beta$; therefore $C = 0$, and

$$\theta = r \operatorname{cosec} \beta \cos^{-1} (\sec \beta \cos \phi) - r \sin \beta \cos^{-1} (\tan \beta \cot \phi)$$

is the equation to the road, where $r = \frac{a^2 + b^2}{2b}$.

$$\text{Integrating (2),} \quad \lambda = -\frac{r \cos \phi}{\sin \beta} + C.$$

When $\lambda = 0$, $\phi = \beta$; therefore $C = \frac{r \cos \beta}{\sin \beta}$, and $\lambda = \frac{r (\cos \beta - \cos \phi)}{\sin \beta}$.

When $\phi = \delta$, $\lambda = \frac{r (\cos \beta - \cos \delta)}{\sin \beta} = r \cot \beta - (r - b) \operatorname{cosec} \beta$,

the length of the road.

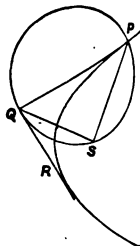
When the hill is a hemisphere, $b = r$ and $\lambda = r \cot \beta$

6976. (By J. YOUNG, B.A.)—Draw a common tangent to a parabola and any circle intersecting it and passing through the focus of the parabola.

*Solution by the PROPOSER; J. O'REGAN;
and others.*

At a point P where the circle cuts the parabola, draw a tangent to the parabola; at the point Q where this tangent again meets the circle, draw a tangent to the circle. This line will be a common tangent, for, if S be the focus, $\angle SQR = \angle SPQ$ from the tangent property of the circle; and this is the condition that QR should be a tangent to the parabola.

As the circle cuts the parabola in two points, these are two real common tangents.



6600. (By Professor SMITZ, M.A.)—A plane triangle is formed by joining three points taken at random in a given semi-circumference; show that the average area (1) of the triangle is $\frac{3r^2}{\pi} \left(1 - \frac{8}{\pi^2}\right)$, and (2) of its inscribed circle is $\pi r^2 \left(\frac{5}{2} - \frac{162}{\pi^2} + \frac{432}{\pi^3}\right)$.

Solution by the PROPOSER.

Let MCN be the given semi-circumference, O its centre, A, B, C the three random points, and Q the centre of the inscribed circle of ABC.

Let $OM = r$, $QR = x$, arc $AM = r\theta$,
arc $AB = r\phi$, arc $AC = r\psi$, and area $ABC = u$.
Then $AB = 2r \sin \frac{1}{2}\phi$, $AC = 2r \sin \frac{1}{2}\psi$,

$BR = 4r \cos \frac{1}{4}\phi \cos \frac{1}{4}\psi \sin \frac{1}{4}(\phi - \psi)$, $u = 2r^2 \sin \frac{1}{2}\phi \sin \frac{1}{2}\psi \sin \frac{1}{2}(\phi - \psi)$,
and $x = BR \tan \frac{1}{4}\psi = 4r \cos \frac{1}{4}\phi \sin \frac{1}{4}\psi \sin \frac{1}{4}(\phi - \psi)$.

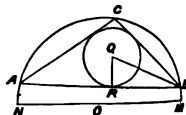
The limits of θ are 0 and π ; of ϕ , 0 and θ ; and of ψ , 0 and ϕ .

1. The average area of the triangle is

$$\frac{\int_0^\pi \int_0^\theta \int_0^\phi u r d\theta r d\phi r d\psi}{\int_0^\pi \int_0^\theta \int_0^\phi r d\theta r d\phi r d\psi} = \frac{6}{\pi^3} \int_0^\pi \int_0^\theta \int_0^\phi u d\theta d\phi d\psi$$

$$= \frac{6r^2}{\pi^3} \int_0^\pi \int_0^\theta \int_0^\phi (1 - \cos \phi - \frac{1}{2}\phi \sin \phi) d\theta d\phi = \frac{3r^2}{\pi^3} \int_0^\pi (2\theta + \theta \cos \theta - 3 \sin \theta) d\theta$$

$$= \frac{3r^2}{\pi^3} \left(1 - \frac{8}{\pi^2}\right).$$



2. The average area of the inscribed circle is

$$\begin{aligned} \frac{6}{\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \pi x^2 d\theta d\phi d\psi &= \frac{12x^2}{\pi^2} \int_0^{\pi} \int_0^{\pi} (2\phi + \phi \cos \frac{1}{2}\phi - 6 \sin \frac{1}{2}\phi) \cos^2 \frac{1}{2}\phi d\theta d\phi \\ &= \frac{3x^2}{2\pi^2} \int_0^{\pi} (5\theta^2 + 24\theta \sin \frac{1}{2}\theta + 2\theta \sin \theta - 110 + 96 \cos \frac{1}{2}\theta + 14 \cos \theta) d\theta \\ &= \pi x^2 \left(\frac{5}{2} - \frac{162}{\pi^2} + \frac{432}{\pi^3} \right). \end{aligned}$$

3083 & 5143. (By Dr. JAMES MATTHEW.)—Find a common value of x that will make $x^2 \pm ax = \square$, $x^2 \pm bx = \square$, $x^2 \pm cx = \square$, &c., *ad inf.*; and again $a^2x^2 \pm ax = \square$, $b^2x^2 \pm bx = \square$, $c^2x^2 \pm cx = \square$, &c., *ad inf.*

Solution by Dr. D. H. HAWK.

(3083). In order to solve this problem, it will be necessary to premise the following theorems:—(1) Every prime number of the form of $4n+1$ is the sum of two squares in one way only; (2) the square of every such number is also the sum of two squares in one way only; (3) the product of any two such numbers is the sum of two squares in two ways, the product of any three such numbers is the sum of two squares in four ways, the product of any four such numbers is the sum of two squares in eight ways, and, in general, the product of any (n) such numbers is the sum of two squares in 2^{n-1} ways. By attending to these theorems, the following solution will be readily understood.

Let $x = z^2$, then, by substitution and expanding z^2 , we have $z^2 \pm a = \square$, $z^2 \pm b = \square$, $z^2 \pm c = \square$, &c. *ad infinitum*.

Let $z^2 = m^2 + n^2$, and $a = 2mn$, then $z^2 \pm a = m^2 \pm 2mn + n^2 = (m \pm n)^2$, also, let $z^2 = p^2 + q^2$, and $b = 2pq$, then $z^2 \pm b = p^2 \pm 2pq + q^2 = (p \pm q)^2$; again, let $z^2 = r^2 + s^2$, and $c = 2rs$, then $z^2 \pm c = r^2 \pm 2rs + s^2 = (r \pm s)^2$, and so on, *ad infinitum*. Here all the conditions of the problem will be satisfied, if we can divide z^2 into two squares in any (n) number of ways. In order to do this, it will be necessary to solve the following problem: "To divide a given square number into two other square numbers." Let $A^2 = z^2$ be the given square, and v^2, w^2 be the required squares; then $A^2 = v^2 + w^2$, and $A^2 - w^2 = v^2$. Assume $A + w = \frac{p'r}{q'}$, and $A - w = \frac{q'o}{p'}$.

Hence, we readily find

$$2A = \frac{p'r}{q'} + \frac{q'o}{p'} = \frac{(p'^2 + q'^2) \cdot v}{p'q'}, \text{ and } 2w = \frac{p'r}{q'} - \frac{q'o}{p'} = \frac{(p'^2 - q'^2) \cdot v}{p'q'};$$

whence
$$r = \frac{2p'q'A}{p'^2 + q'^2} \text{ and } w = \frac{(p'^2 - q'^2)A}{p'^2 + q'^2};$$

in which formulæ A , or its equal z , may be taken any prime number or product of two or more prime numbers, each of which is the sum of two squares; then $p^2 + q^2$ being put successively equal to each sum of two squares into which A or z can be divided, we shall have as many integral values of v, w ; whence also $z^2 = v^2 + w^2$ the same number of different ways, for which may be put successively $m^2 + n^2, p^2 + q^2, r^2 + s^2$, &c. The different values of $2vw$ may be put successively

$$= 2mn, 2pq, 2rs, \&c. = a, b, c, \&c.$$

Let $A = z = 5 = 2^2 + 1^2$, and put $p' = 2, q' = 1$, then $v = 4, w = 3$, and $z^2 = 5^2 = 4^2 + 3^2$, being the sum of two squares in *one* way only. Let $A = z = 5 \cdot 13 = 65$, then

$$5 = 2^2 + 1^2, 13 = 3^2 + 2^2, 65 = 8^2 + 1^2 = 7^2 + 4^2,$$

and, putting p', q' successively = the roots of each set of two squares, we shall have $v = 52, w = 39$; $v = 60, w = 25$; $v = 16, w = 63$; $v = 56, w = 33$; whence z^2 = sum of two squares in *four* ways. Let

$$z = 5 \cdot 13 \cdot 17 = 1105,$$

then

$$5 = 2^2 + 1^2, 13 = 3^2 + 2^2, 17 = 4^2 + 1^2;$$

$$5 \cdot 13 = 65 = 8^2 + 1^2 = 7^2 + 4^2; 5 \cdot 17 = 85 = 9^2 + 2^2 = 7^2 + 6^2;$$

$$13 \cdot 17 = 221 = 11^2 + 10^2 = 14^2 + 5^2;$$

and

$$1105 = 33^2 + 4^2 = 32^2 + 9^2 = 31^2 + 12^2 = 24^2 + 23^2;$$

and, putting p', q' successively = the roots of each set of two squares, we shall have the following values of v, w , viz.: 884, 663; 1020, 425; 520, 975; 272, 1071; 952, 561; 468, 1001; 1092, 169; 1100, 105; 700, 855; 264, 1073; 576, 943; 744, 817; 1104, 47; whence z^2 = sum of two squares in *thirteen* ways.

In like manner, if z = the product of four factors, each of which is the sum of two squares; then, taking the factors singly, the product of every two, the product of every three, and finally, the product of all four, we shall find *forty* ways in which z^2 will be divided into the sum of two squares. And, if z = the product of five such factors, we shall, by proceeding as above, find *one hundred and twenty-one* ways in which z^2 will be divided into the sum of two squares, and so on for 6, 7, 8... to n factors. Whence we get the series 1, 4, 13, 40, 121, 364, 1093, &c. the law of continuation of which is $3m' + 1$, m' being any term in the series. Since $1 = 1, 4 = 3 + 1, 13 = 3^2 + 3 + 1, 40 = 3^3 + 3^2 + 3 + 1$, &c., it is evident that the n^{th} term of the series is the sum of a geometric series, whose first term is 1, ratio 3, and last term 3^{n-1} ; therefore $S = \text{sum} = \frac{1}{2} (3^n - 1)$, where n is the number of factors used in the n^{th} term.

Having thus found the values of z^2, a, b, c , &c. in the expressions $x^2 \pm a = \square, x^2 \pm b = \square, x^2 \pm c = \square$, &c., if these expressions be multiplied by $x = z^2$, we shall solve the original expressions $x^2 \pm ax = \square, x^2 \pm bx = \square, x^2 \pm cx = \square$, &c.

(5143). This problem can be solved in the same manner as Quest. 3083; for, if the expressions be divided by a^2, b^2, c^2 , &c. respectively, we shall have

$$x^2 \pm \frac{1}{a} x = \square, x^2 \pm \frac{1}{b} x = \square, x^2 \pm \frac{1}{c} x = \square, \&c. \text{ ad infinitum};$$

and, putting $a' = \frac{1}{a}$, $b' = \frac{1}{b}$, $c' = \frac{1}{c}$, &c.,

the expressions become

$$x^2 \pm a'x = \square, \quad x^2 \pm b'x = \square, \quad x^2 \pm c'x = \square, \quad \&c. \text{ ad. inf.},$$

which are of the same form as those in 3083, and may be solved in the same way; then, a' , b' , c' , &c. being known, we have

$$a = \frac{1}{a'}, \quad b = \frac{1}{b'}, \quad c = \frac{1}{c'}, \quad \&c.$$

4722. (By Professor CAYLEY, F.R.S.)—1. Show that the conditions in regard to the reality of the roots of the equations $(x^2 - a)^2 + 16A(x - m) = 0$, are, if

$$(4m^2 - 3a)^3 - (8m^3 - 9ma - 27A)^2 = -,$$

then the roots are two real, two imaginary; but if

$$(4m^2 - 3a)^3 - (8m^3 - 9ma - 27A)^2 = +,$$

then, if simultaneously $a = +$, $A(ma - 9A) = +$,

the roots are all real, but otherwise they are all imaginary.

2. If the roots of the foregoing equation are all imaginary, then for any real value whatever of y , the roots of the equation

$$(x^2 + y^2 - a)^2 + 16A(x - m) = 0$$

are all imaginary.

Solution by W. J. C. SHARP, M.A.

The invariants of $(x^2 - a)^2 + 16A(x - m) = 0$,

or $x^4 - 2ax^2 + 16Ax + a^2 - 16Am$,

are $S = \frac{1}{3}(a^3 - 12Am)$ and $T = -\frac{1}{27}[a^3 - 18Ama + 54A^2]$,

and the discriminant

$$S^2 - 27T^2 = \frac{2}{27}A^2[(4m^2 - 3a)^3 - (8m^3 - 9ma + 27A)^2],$$

and therefore (SALMON'S *Higher Algebra*, p. 175) the roots are two real and two imaginary if the quantity in the bracket be negative; and all real if that quantity be positive, and

$$(b^2 - ac) \equiv \frac{1}{3}a \text{ and } 3aT + 2(b^2 - ac)S \equiv \frac{1}{9}A(ma - 9A)$$

be positive; and all imaginary if the quantity in the bracket be positive, and either $\frac{1}{3}a$ or $A(ma - 9A)$ negative; and if $\frac{1}{3}a$ be negative, the quantity will continue so when $a - y^2$ is written for a , which proves (2).

The form $(x^2 + y^2 - a)^2 + 16A(x - m) = 0$ is that to which Professor CAYLEY has reduced the equation to a Cartesian; and hence it appears that those values of y which make $(4m^2 + 3y^2 - a)^3 - (8m^3 - 9am + 9my^2 + 27A)^2$ negative, correspond to two real and two imaginary values of x , whilst those which make the same quantity positive, give four real values of x if $a - y^2$ and $A(ma - my^2 - 9A)$ be both positive, and otherwise none.

7249. (By Professor SYLVESTER, F.R.S.)—If

$$1 - \frac{1}{1-c} y + \frac{c}{(1-c)(1-c^2)} y^2 + \dots \pm \frac{c^{i(n^2-n)}}{(1-c)(1-c^2) \dots (1-c^n)} y^n + \dots = 0,$$

prove that (1) the sum of the i^{th} powers of the several values of $\frac{1}{y}$ is $\frac{1}{1-c^i}$; and (2) the sum of the i^{th} homogeneous products of the same values is

$$\frac{1}{(1-c)(1-c^2) \dots (1-c^i)},$$

[If $i = 2$, the first sum is $\left(\frac{1}{1-c}\right)^2 - 2 \frac{c}{(1-c)(1-c^2)}$, which is $\frac{1}{1-c^2}$ and the second sum is $\left(\frac{1}{1-c}\right)^2 - \frac{1}{(1-c)(1-c^2)}$, which is $\frac{1}{(1-c)(1-c^2)}$.]

Solution by the Rev. T. R. TERRY, M.A.; G. HEPFEL, M.A.; and others.

By Algebra it may be shown that, if

$$A_n = \frac{c^{n-1}}{1-c^n} A_{n-1}, \text{ and } B_n = \frac{1}{1-c^n} B_{n-1},$$

$$(1-y)(1-cy)(1-c^2y) \dots = 1 - A_1y + A_2y^2 - A_3y^3 \dots \quad \dots (1),$$

$$\text{and } \frac{1}{(1-y)(1-cy)(1-c^2y) \dots} = 1 + B_1y + B_2y^2 + \dots \quad \dots (2).$$

From (1) it follows that the given equation may be put in the form

$$(1-y)(1-cy)(1-c^2y) \dots = 0;$$

hence sum of n^{th} powers is $1 + c^n + c^{2n} + \dots = \frac{1}{1-c^n}$; and the sum of the n^{th} homogeneous products = $B_n = \frac{1}{(1-c)(1-c^2) \dots (1-c^n)}$.

[If the equation stop with the term involving y^n , the values of the series of the powers and of the sums of the homogeneous products will be unaltered up to n dimensions.]

6856. (By Professor TOWNSEND, M.A.)—In the irrotational strain of an incompressible lamina in the plane of its mass, if the equations $r^n \cos n\theta = a^n$ and $r^n \sin n\theta = b^n$, where a and b are parameters, represent the line-systems of greatest and least displacement of the molecules; show that the equations $r^{4n} \cos \frac{1}{2}n\theta = a^{4n}$ and $r^{4n} \sin \frac{1}{2}n\theta = b^{4n}$, where a and b are parameters, represent the line-systems of greatest and least dilatation of the strain.

Solution by the PROPOSER.

Since generally, when in rectangular coordinates x and y , the equations $u = a$ and $v = b$, where u and v are conjugate functions of x and y , and where a and b are arbitrary parameters, represent in finite terms the line

systems of greatest or least molecular displacement in a strain such as that of the question, the differential equations

$$\frac{d^2u}{dx dy} \cdot \frac{dy}{dx} + \frac{d^2u}{dx^2} \pm \left[\left(\frac{d^2u}{dx dy} \right)^2 + \left(\frac{d^2u}{dx^2} \right)^2 \right]^{\frac{1}{2}} = 0,$$

or their equivalents

$$\frac{d^2v}{dy^2} \cdot \frac{dy}{dx} + \frac{d^2v}{dx dy} \pm \left[\left(\frac{d^2v}{dy^2} \right)^2 + \left(\frac{d^2v}{dx dy} \right)^2 \right]^{\frac{1}{2}} = 0,$$

represent, as is well known, the line systems of greatest and least dilatation of the strain; hence, when in polar coordinates $u = k \cdot r^n \cos n\theta$ and $v = k \cdot r^n \sin n\theta$, where k is a constant of the order of magnitude of the displacements determining the strain, since then

$$\frac{d^2u}{dx^2} = -\frac{d^2u}{dy^2} = \frac{d^2v}{dx dy} = +n(n-1) \cdot k \cdot r^{n-2} \cos(n-2)\theta,$$

$$\frac{d^2v}{dy^2} = -\frac{d^2v}{dx^2} = \frac{d^2u}{dx dy} = -n(n-1) \cdot k \cdot r^{n-2} \sin(n-2)\theta,$$

the aforesaid differential equations, after substitution for dy and dx of their equivalents $\sin \theta dr + r \cos \theta d\theta$ and $\cos \theta dr - r \sin \theta d\theta$, become in polar coordinates

$$[\cos(n-1)\theta \pm \cos \theta] dr - [\sin(n-1)\theta \pm \sin \theta] r d\theta = 0,$$

or, for the upper and lower signs taken separately,

$$\frac{dr}{r} - \frac{\sin \frac{1}{2}n\theta d\theta}{\cos \frac{1}{2}n\theta} = 0 \quad \text{and} \quad \frac{dr}{r} + \frac{\cos \frac{1}{2}n\theta d\theta}{\sin \frac{1}{2}n\theta} = 0,$$

and as the complete integrals of the latter infinite terms are respectively.

$$r^{1n} \cos \frac{1}{2}n\theta = a^{1n} \quad \text{and} \quad r^{1n} \sin \frac{1}{2}n\theta = b^{1n},$$

where a and b are the two constants of integration, therefore, &c.

Since evidently, from the aforesaid values of u and v ,

$$\left[\left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right]^{\frac{1}{2}} = \left[\left(\frac{dv}{dx} \right)^2 + \left(\frac{dv}{dy} \right)^2 \right]^{\frac{1}{2}} = nr^{n-1},$$

$$\left[\left(\frac{d^2u}{dx dy} \right)^2 - \frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} \right]^{\frac{1}{2}} = \left[\left(\frac{d^2v}{dx dy} \right)^2 - \frac{d^2v}{dx^2} \cdot \frac{d^2v}{dy^2} \right]^{\frac{1}{2}} = n(n-1)r^{n-2},$$

therefore, in the strain forming the subject of the question, the molecular displacement and principal dilatation depend throughout the mass only on the distance from the centre of the strain, and vary respectively as the $(n-1)^{\text{th}}$ and as the $(n-2)^{\text{th}}$ powers of that distance.

7255. (By Professor MARTIN, M.A., Ph.D.)—An urn contains 12 balls of which either 5, or 6, or 7, or 8, are white (but which one of these numbers is not known), and the rest are black; if a ball is drawn at random, find the chance that it is white.

Solution by the Rev. T. R. TERRY, M.A.; E. RUTTER; and others.

We assume that it is equally likely that there will be 5 or 6, or 7 or 8, white balls in the urn, and the chance required is obviously

$$= \frac{1}{4} \left(\frac{5+6+7+8}{12} \right) = \frac{13}{24}.$$

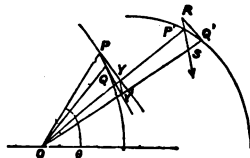
7259. (By Professor MINCHIN, M.A.) — A point P describes an orbit, with acceleration f , directed always to a fixed centre O; prove that the acceleration of the corresponding point P', of the Hodograph, is in general, non-central, and makes with the radius vector OP' an angle

$$\chi \text{ given by the equation } \tan \chi = \frac{\frac{d}{d\theta} \log \frac{f}{r}}{1 - \cot \phi \frac{d}{d\theta} \log f}, \text{ where } \phi \text{ is the angle}$$

between OP and the tangent at P, $r = OP$, and θ is the polar angle of r .

Solution by J. J. WALKER, M.A.

Let PQ be an element of the path, P'Q' the corresponding element of the hodograph, regarded as the inverse of the pedal, OY, OY' being perpendiculars on the tangents at P, Q. Q'P' representing the velocity at P', let Q'R represent that at Q', then RP' is the direction of the acceleration at P', and



$$OP'S = \chi = (OP'Q' \text{ or } OY'Y \text{ or}) \phi - Q'P'S;$$

hence
$$\tan \chi = \frac{\cot Q'P'S - \cot \phi}{\cot \phi \cot Q'P'S + 1}.$$

But $\cot Q'P'S = \frac{Q'R - Q'P'}{Q'R d\theta} = -\frac{d \log f}{d\theta}$; and $\cot \phi = -\frac{dr}{r d\theta} = -\frac{d \log r}{d\theta}$

Thus
$$\tan \chi = -\frac{\frac{d \log \frac{f}{r}}{d\theta}}{1 - \cot \phi \frac{d \log f}{d\theta}}$$

the sign depending on the convention in taking χ .

The acceleration (a) in the hodograph can be deduced thus:—

$$RP' = a dt = Q'R \frac{d\theta}{\sin Q'P'S}, \text{ or } a \propto \frac{f d\theta}{\sin P'dt} \propto \frac{f}{r^2} \left\{ 1 + \left(\frac{d \log f}{d\theta} \right)^2 \right\}^{\frac{1}{2}},$$

where $r = OP$.

7060. (By R. TUCKER, M.A.)—Three parabolas are described about the triangle ABC, so that A is the vertex of the diameter bisecting BC, and so on; show that (1) if p, q, r are the respective parameters of the diameters, then $\frac{1}{4}[p^{-2} + q^{-2} + r^{-2}] = 2\sum a^2 \cdot \sum a^{-4} - 3\sum a^{-2} \dots (\alpha)$,
 $a^2 p^{-2} + \dots = 12(a^2 + b^2 + c^2)$, $a^2 p^{-4} + \dots = 144(a^4 + b^4 + c^4) \dots (\beta, \gamma)$;
 (2) when ABC is an equilateral triangle, the area of the triangle formed by joining the foci = $\frac{1}{8}\Delta ABC$.

Solution by R. KNOWLES, B.A.; J. HAMMOND, M.A.; and others.

1. Here $p \cdot AD = DC^2$, therefore

$$p^2 = \frac{a^4}{8(b^2 + c^2 - \frac{1}{2}a^2)},$$

and similarly for q^2 and r^2 . Therefore

$$\frac{1}{4}(p^{-2} + q^{-2} + r^{-2}) = \frac{2}{a^4 b^4 c^4} [b^4 c^4 (b^2 + c^2 - \frac{1}{2}a^2) + a^4 c^4 (a^2 + c^2 - \frac{1}{2}b^2) + a^4 b^4 (a^2 + b^2 - \frac{1}{2}c^2)],$$

which is identically the same as

$$2(a^2 + b^2 + c^2) \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} \right) - 3 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right),$$

which proves (α); and (β) and (γ) may be proved similarly.

2. The triangle ABC being equilateral, the lines AD, &c. are the axes of the parabolas and A, B, C the vertices. Therefore

$$4O_1A \times AD = DC^2, \text{ therefore } O_1A = O_2B = O_3C = \frac{1}{2\sqrt{3}}a,$$

and the coordinates of O_1, O_2, O_3 are respectively

$$\left(\frac{1}{2}a, \frac{1}{2\sqrt{3}}a\right), \left(\frac{1}{2}a, -\frac{1}{2\sqrt{3}}a\right), \left(\frac{1}{2}a, \frac{1}{2\sqrt{3}}a\right),$$

$$\text{and } 2\Delta O_1O_2O_3 = \frac{1}{2\sqrt{3}}a^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) + \frac{1}{2\sqrt{3}}a^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) + \frac{1}{2\sqrt{3}}a^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = \frac{1}{12\sqrt{3}}a^2 \left(\frac{1}{2} + \frac{1}{2} - 1\right) = \frac{1}{24}a^2 \cdot \frac{1}{\sqrt{3}}a^2,$$

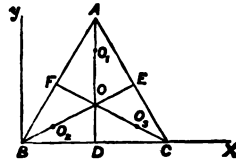
therefore

$$\Delta O_1O_2O_3 = \frac{1}{8}\Delta ABC.$$

[Otherwise: $OA = \frac{2}{3}AD = \frac{1}{3}a\sqrt{3} = \frac{1}{\sqrt{3}}a\sqrt{3}$; therefore

$$OO_1 = \frac{1}{\sqrt{3}}a\sqrt{3} - \frac{1}{2\sqrt{3}}a\sqrt{3} = \frac{1}{2\sqrt{3}}a\sqrt{3}, \text{ therefore } \frac{OO_1}{OA} = \frac{1}{2};$$

and the triangles $O_1O_2O_3, ABC$ are similar, therefore $\Delta O_1O_2O_3 = \frac{1}{8}\Delta ABC$.]



6221. (By Professor WOLSTENHOLME, M.A.)—Trace the curve

$$\frac{12a}{\pi} \int_0^\infty \frac{\sin x \sin 2x \sin 3x \sin 4x}{x^4} \cos \frac{xz}{a} dx,$$

and prove that its area (between the curve and the axis of x) is $288a^2$.

Solution by the PROPOSER.

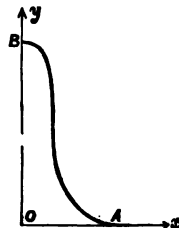
The figure is as annexed; and the values of $\frac{y}{a}$ are

$$\frac{1}{2} \left\{ \left(8 - \frac{x}{a} \right)^2 \left(4 + \frac{x}{a} \right) + 16 \right\}, \text{ if } \frac{x}{a} > 0 < 6;$$

$$\frac{1}{2} \left\{ 3 \left(9 - \frac{x}{a} \right)^2 + 1 \right\}, \text{ if } \frac{x}{a} > 6 < 8;$$

$$\frac{1}{2} \left(10 - \frac{x}{a} \right)^2, \text{ if } \frac{x}{a} > 8 < 10;$$

and 0, if $\frac{x}{a} > 10$.



The coordinates of the centroid of the area on the positive side of the axis of x are $\bar{x} = \frac{1698}{100}a = 3.607a$, $\bar{y} = \frac{1722}{100}a = 12.45a$.

Also for any negative value of x , y has the same value as for the same positive value. The shape of one half the curve is as in the figure. OA = 10a, OB = 34a (rather too small in figure).

7201. (By R. F. SCOTT, M.A.)—Prove that

$$\int_0^{\pi} x \log (1 - \sin^2 \alpha \sin^2 x) dx = 2\pi^2 \log \cos \frac{1}{2} \alpha.$$

Solution by R. KNOWLES, B.A.; C. MORGAN, M.A.; and others.

Given integral $I = \int_0^{\pi} \log (1 - \sin^2 \alpha \sin^2 x) dx = \pi J$ say;

$$\begin{aligned} \text{then } \frac{dJ}{d\alpha} &= -2 \sin \alpha \cos \alpha \int_0^{\pi} \frac{\sin^2 x}{1 - \sin^2 \alpha \sin^2 x} dx \\ &= -2 \sin \alpha \cos \alpha \left(\frac{1}{2} \cdot \frac{1}{2} \pi + \sin^2 \alpha \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{2} \pi + \dots \right) \\ &= -\pi \frac{\cos \alpha}{\sin \alpha} \left(\frac{1}{\cos \alpha} - 1 \right) = -\pi \tan \frac{1}{2} \alpha, \end{aligned}$$

therefore $J = 2\pi \log \cos \frac{1}{2} \alpha + C$.

To find C, put $\alpha = \frac{1}{2} \pi$, therefore $2 \log \cos x = \pi \log \frac{1}{2} + C$, therefore $C = 0$, therefore $I = 2\pi^2 \log \cos \frac{1}{2} \alpha$.

6933. (By J. J. WALKER, M.A.)—A homogeneous rectangular parallelepiped, the edges of which are a, b, c , floats in a liquid (the ratio of the densities of the two being $\rho : 1$) with the edge a vertical; show that,

if displaced through an angle θ (the top remaining above the surface of the liquid), so that the plane of ab remains parallel to itself, the body will right itself if $a^2 + b^2$ is less than $(1 + \sec^2 \theta) + 12\rho(1 - \rho)$; and hence find the condition of stability for small displacements.

Solution by D. EDWARDS; PROFESSOR NASH, M.A.; and others.

Let the plane of floatation in the inclined position of the solid cut the vertical plane through the centroid in the line AB . Let C, G_1 be the centroids of the body and of the immersed portions respectively, G that of the rectangle $ALMD$, and g that of the triangle ADB . Let $MN (= h)$ be the length of axis which is immersed when the solid is vertical. Let O be the middle point of AB , and let CR, G_1S be drawn vertically. The solid will right itself if $SG_1 > CR$. Now

$$GC = \frac{1}{2}a - \frac{1}{2}h + \frac{1}{2}b \tan \theta,$$

$$OC = h - \frac{1}{2}a, \text{ also } \angle GCR = \theta, \text{ whence } CR = \frac{b(a - h + \frac{1}{2}b \tan \theta)}{6h \sin \theta + b(1 + \cos^2 \theta) \sec \theta}.$$

$$\text{Again, } GG : G_1g = b \tan \theta : 2h - b \tan \theta,$$

$$\text{whence } G_1S = \frac{b}{2h} Og = \frac{b^2}{12h} \sec \theta.$$

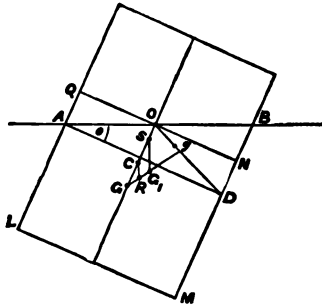
$$\therefore b(a - h + \frac{1}{2}b \tan \theta) \cos \theta > \frac{b^2}{12h} [6h \sin \theta + b(1 + \cos^2 \theta) \sec \theta],$$

$$\text{or } 12h(a - h) \cos^2 \theta < b^2(1 + \cos^2 \theta),$$

$$\text{or, since } h = a\rho, \quad 12a^2\rho(1 - \rho) < b^2(1 + \sec^2 \theta).$$

When θ is small, this condition becomes $6a^2\rho(1 - \rho) < b^2$; and, if Al^2 be the moment of inertia of the section bc about a line through O perpendicular to the vertical plane through the centroid, V the volume of water displaced, and z the distance between the centroid of solid and portion immersed, this is of course equivalent to $Al^2 > Vz$.

The distances, SG_1, CR are investigated quite independently of one another, and therefore, *contrary to what at first sight might appear from inspection of the figure*, it is not assumed that SG_1 is either greater or less than CR .



7208. (By A. McMURCHY, B.A.)—If n be the length of a normal chord to a surface $ax^2 + by^2 + cz^2 = 1$ at a point for which r is the distance from the centre, and p the perpendicular on the tangent plane, prove that

$$\frac{2}{np^3} - \frac{a+b+c}{p^2} + bc + ca + ab - abc r^2 = 0.$$

Solution by the Rev. T. R. TERRY, M.A.; SARAH MARKS; and others.

With the given notation, we have

$$\frac{1}{r^2} = a^2x^2 + b^2y^2 + c^2z^2, \quad \gamma^2 = x^2 + y^2 + z^2, \quad 1 = ax^2 + by^2 + cz^2 \dots (1, 2, 3).$$

To find α , we have

$$\frac{\xi - x}{ax} = \frac{\eta - y}{by} = \frac{\zeta - z}{cz} = \alpha p, \quad \text{therefore} \quad \xi = x(1 + \alpha p),$$

therefore, substituting in the equation to the surface, we get

$$\alpha p = \frac{2(a^2x^2 + b^2y^2 + c^2z^2)}{a^2y^2 + b^2y^2 + c^2z^2}, \quad \text{or} \quad \frac{2}{\alpha p^2} = a^2x^2 + b^2y^2 + c^2z^2 \dots (4).$$

Eliminating x^2, y^2, z^2 from (1, 2, 3), (4), we obviously have the required result.

6691. (By Professor HUDSON, M.A.)—Prove (1) that, if the curve $r = a + b \cos \theta$ have a point of inflexion, a must lie between b and $2b$; and (2) trace the change in form of the curve, as a changes from 0 to $2b$.

Solution by Professor SCOTT, M.A.; J. O'REGAN; and others.

The curve is the limaçon, the inverse of a conic with regard to the focus. If $a = 0$, the curve is a circle with diameter b ; if $a = b$, it becomes the Cardioid; if $a < b$, it resembles the Cardioid with a loop inside the curve, which loop disappears when $a > b$. Forming the expression

$$0 = r^3 + 2 \left(\frac{dr}{d\theta} \right)^2 - \frac{r d^2r}{d\theta^2} = a^2 + 3ab \cos \theta + 2b^2,$$

$$\text{we get} \quad -\cos \theta = \frac{a^2 + 2b^2}{3ab};$$

$$\text{whence} \quad a^2 + 2b^2 < 3ab, \quad \text{or} \quad a^2 + b^2 - 3ab \equiv (a - b)(a - 2b) < 0,$$

$$\text{whence} \quad a > b < 2b.$$

7197. (By the EDITOR)—With two given sides, construct a triangle such that the square of the sine of half the angle contained by the sides may be equal to the squares of the sines of the other two semi-angles.

I. Solution by W. H. BLYTHE, M.A.; SARAH MARKS; and others.

Let a, b , be the given sides of the triangle, and let

$$\sin^2 \frac{1}{2}C = \sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B; \quad \text{then} \quad 1 + \cos C = \cos A + \cos B,$$

$$\therefore 2 \cos^2 \frac{1}{2}C = 2 \cos \frac{1}{2}A \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B), \quad \therefore \cos \frac{1}{2}(A - B) = \cot \frac{1}{2}C \cos \frac{1}{2}C;$$

and, as $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{1}{2}C$, $\therefore \tan^2 \frac{1}{2}C \sec^2 \frac{1}{2}C = 1 + \left(\frac{a-b}{a+b}\right)^2 \cot^2 \frac{1}{2}C$,

therefore $\tan^6 \frac{1}{2}C + \tan^4 \frac{1}{2}C - \tan^2 \frac{1}{2}C = \frac{(a-b)^2}{(a+b)^2}$.

Hence, putting $3 \tan^2 \frac{1}{2}C = 4 \cos \phi - 1$, we have

$$(a+b)^2 (16 \cos 3\phi + 11) = 27 (a-b)^2;$$

whence we find 3ϕ , then ϕ , and from ϕ , $\frac{1}{2}C$; and the problem reduces to ordinary solution of triangles, a , b , and C given.

[See the solutions of the EDITOR'S QUEST. 6287, on pp. 35—38 of Vol. 36 of the *Reprints*, where a construction is given by the aid of a parabola and a rectangular hyperbola.]

II. Solution by M. COLLINS, B.A.; J. O'REGAN; and others.

The triangle is of the form in part (3) of Quest. 6287. Thus, if c be the third or required side, Q the centre, r the radius of the inscribed circle, and O the centre of the circumscribed circle, we have $r(a+b+c) = ab \sin C$, which differentiated gives (since r is a maximum, and therefore $dr = 0$)

$$r dc = ab \cos C \cdot dC;$$

also $c^2 = a^2 + b^2 - 2ab \cos C$;

whence, differentiating,

$$c \cdot d = ab \sin C \cdot dC;$$

and from these equations we obtain

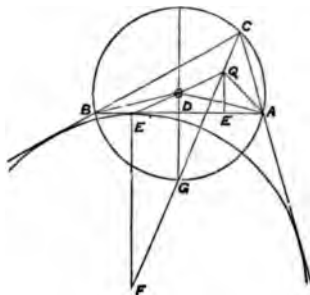
$$r = c \cot C = 2BD \cdot \cot BOD = 2OD;$$

hence $DE = DE'$, therefore $E'F = 2OG$, that is, r' = radius of circle escribed to side $c = 2R =$ diameter of circumscribed circle.

Now $R = \frac{abc}{4\Delta}$, and $r'(s-c) = \Delta$, therefore $2(s-c) = \frac{4\Delta^2}{b}$,

therefore $abc(s-c) = 2\Delta^2 = 2s(s-a)(s-b)(s-c)$,

and as $c = 2s - a - b$, we get $2s^2 - 2(a+b)s^2 + ab(a+b) = 0$, an equation which determines s , and therefore the construction of the triangle. It is easy to see that the line DQ , which joins the middle D of the base to the centre Q of the inscribed circle, would pass through the point of intersection of the three altitudes of the required $\triangle ABC$ when its inscribed circle (or its radius QE) is a maximum.



7224. (By Professor COCHEZ.)—Démontrer que la fonction

$$ke^{2x} \{ [(x+y)^2 + c]^{\frac{1}{2}} - x - y \} = [(x+y)^2 + c]^{\frac{1}{2}} + x + y,$$

est telle que $x + y = y''$.

Solution by the Rev. J. L. KITCHIN, M.A. ; D. EDWARDS ; and others.

We have
$$\frac{[(x+y)^2+c]^{\frac{1}{2}}+x+y}{[(x+y)^2+c]^{\frac{1}{2}}-x-y} = ke^{2x},$$

$$\therefore \frac{(x+y)^2+c}{(x+y)^2} = \frac{(ke^{2x}+1)^2}{(ke^{2x}-1)^2} \quad \therefore x+y = \left(\frac{c}{4k}\right)^{\frac{1}{2}} (ke^x - e^{-x}),$$

therefore
$$y' = x+y.$$

6864. (By Professor HUDSON, M.A.)—If I, O be the centres of the inscribed and circumscribed circles of a spherical triangle, r, R their radii, and $2s = a + b + c$, prove that $2 \sin s \cos IO = (\sin a + \sin b + \sin c) \cos r \cos R$.

Solution by G. M. REEVES, M.A. ; Professor MATZ, M.A. ; and others.

Let O be the centre of the circumscribed, and I the centre of the inscribed circles of a spherical triangle ABC; then

$$\cos OI = \cos OA \cos IA + \sin OA \sin IA \cos OAI.$$

Now $OAC = S - B$ and $IAC = \frac{1}{2}A$,

therefore $OAI = \frac{1}{2}(C - B)$.

Also $OA = R$, and ID perpendicular to AC is equal to r, and $AD = s - a$; therefore, from $\triangle AID$,

$$\cos IA = \cos(s-a) \cos r \quad \text{and} \quad \sin r = \sin IA \sin \frac{1}{2}A,$$

therefore $\cos OI = \cos R \cos r \cos(s-a) + \sin R \frac{\sin r}{\sin \frac{1}{2}A} \cos \frac{1}{2}(B-C)$

$$= \left\{ \cos(s-a) + \frac{\cos \frac{1}{2}(B-C)}{\sin \frac{1}{2}A} \tan R \tan r \right\} \cos R \cos r$$

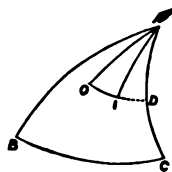
$$= \left\{ \cos(s-a) + \frac{\sin \frac{1}{2}(b+c)}{\sin \frac{1}{2}a} \cdot \frac{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n} \cdot \frac{n}{\sin s} \right\} \times \cos R \cos r,$$

$$\therefore 2 \sin s \cos OI = [2 \sin s \cos(s-a) + 4 \sin \frac{1}{2}(b+c) \sin \frac{1}{2}b \sin \frac{1}{2}c] \cos R \cos r$$

$$= [\sin(b+c) + \sin a + 2 \sin \frac{1}{2}(b+c) \{ \cos \frac{1}{2}(b-c) - \cos \frac{1}{2}(b+c) \}] \cos R \cos r$$

$$= [\sin(b+c) + \sin a + \sin b + \sin c - \sin(b+c)] \cos R \cos r$$

$$= (\sin a + \sin b + \sin c) \cos R \cos r.$$



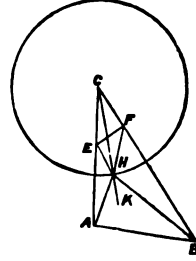
6878. (By T. C. SIMMONS, M.A.)—If A, B are two points in the plane of a given circle; give a geometrical construction for finding the position of a point P on the circumference, such that the lines AP, BP make equal angles with the tangent at P.

Solution by Dr. CURTIS; E. RUTTER; and others.

This, which is known as ALHAZEN's problem, admits of the following geometrical solution:—If C be the centre of the circle and a its radius, take E and F so that $AC \cdot CE = a^2 = BC \cdot CF$; draw the hyperbola-locus of vertex of all triangles whose common base is EF , and the difference of whose base angles $= \angle CEF \sim \angle CFE$; let H be any one of the four points in which this hyperbola cuts the circle; then, as

$$\angle EFH - FEH = \angle CEF - \angle CFE,$$

we have $\angle CEH = \angle CFH$, and in the triangles ACH , HCE , we have $AC : CH = CH : CE$, therefore $\angle AHC = \angle CEH$; similarly $\angle BHC = \angle CFH$, and therefore $\angle AHC = \angle BHC$, and $\angle AHK = \angle BHK$, therefore H is one of the required points.



7215. (By A. H. CURTIS, LL.D., D.Sc.)—Prove that

$$\begin{aligned} \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c. \dots ad\ inf. &= \frac{\pi^4}{90}, \\ \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \&c. \dots ad\ inf. &= \frac{(2^2-1)\pi^4}{2^3 \times 90} = \frac{7\pi^4}{720}, \\ \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \&c. \dots ad\ inf. &= \frac{2^4-1}{2^4} \times \frac{\pi^4}{90} = \frac{\pi^4}{96}; \\ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \&c. \dots ad\ inf. &= \frac{\pi^8}{3 \cdot 5 \cdot 7 \cdot 90} = \frac{\pi^8}{9450}, \\ \frac{1}{1^8} - \frac{1}{2^8} + \frac{1}{3^8} - \&c. \dots ad\ inf. &= \frac{2^7-1}{2^7} \times \frac{\pi^8}{9450} = \frac{127}{128} \times \frac{\pi^8}{9450}, \\ \frac{1}{1^8} + \frac{1}{3^8} + \frac{1}{5^8} + \&c. \dots ad\ inf. &= \frac{2^8-1}{2^8} \times \frac{\pi^8}{9450} = \frac{255}{256} \times \frac{\pi^8}{9450} = \frac{17\pi^8}{256 \times 630}. \end{aligned}$$

Solution by the Rev. T. R. TERRY, M.A.; Professor МУХОМАНДЯН, M.A.; and others.

Since $\frac{\sin \theta}{\theta} = \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{2^2\pi^2}\right) \dots$, we have, by taking logs,

$$\log \left(1 - \frac{\theta^2}{\pi^2} + \dots\right) = - \left(\frac{\theta^2}{\pi^2} + \frac{1}{2} \frac{\theta^4}{\pi^4} + \dots\right) - \left(\frac{\theta^2}{2^2\pi^2} + \dots\right) - \dots$$

Equating coefficients of θ^4 and θ^8 , and denoting the series in the question by $S_1, S_2, \dots$,

$$\begin{aligned} -\frac{1}{2\pi^4} &= \frac{1}{6!} - \frac{1}{2} \frac{1}{3! 3!}, \text{ therefore } S_1 = \frac{\pi^4}{90}, \\ -\frac{1}{4\pi^8} S_4 &= \frac{1}{9!} - \frac{1}{2} \left(\frac{1}{5! 5!} + \frac{2}{3! 7!}\right) + \frac{1}{3} \frac{3}{3! 3! 5!} - \frac{1}{4} \frac{1}{(3!)^4} = -\frac{1}{4} \frac{1}{9450}, \end{aligned}$$

Solution by the Rev. G. RICHARDSON, M.A.

Question 6313 is the trigonometrical equivalent of 7179, which was proposed by me in hope of obtaining a geometrical proof, and so acquiring a proof by elementary geometry of the proposition, that the nine-point circle touches the inscribed circle of a triangle. The following investigation does this, and is of a more elementary character than the proofs usually given of this theorem.

Let O be the orthocentre, S the centre of the circumscribed circle, PSP' the diameter of the latter which bisects BC in A' . Then AP passes through I , the centre of the inscribed circle, and I_1 , the centre of the escribed circle opposite to A . Draw IK (r), I_1m perpendicular to AC and AOD respectively. Then it is easily seen that $\angle ACI_1 = \angle AIB_1$, and consequently that the triangles AIB and ACI_1 are similar; hence we have

$$AI \cdot AI_1 = AB \cdot AC = PP' \cdot AD_1 \text{ (Euc. VI. C.)}$$

i.e., since P bisects II_1 ,

$$2AI \cdot AP - AI^2 = PP' \cdot AD_1$$

$$\therefore 2Am \cdot PP' - AI^2 = PP' \cdot AD_1 \text{ (by sim. } \Delta s.)$$

$$\therefore 2Am = AD + \frac{AI^2}{PP'}.$$

$$\text{Now} \quad IO^2 = AI^2 + AO^2 - 2AO \cdot Am \text{ (Euc. II. 13)}$$

$$= AI^2 + AO^2 - AO \cdot AD - \frac{AO \cdot AI^2}{PP'}$$

$$= AI^2 \cdot \frac{2SP - 2SA'}{PP'} - AO \cdot OD \text{ (since } AO = 2SA')$$

$$= 2AI^2 \cdot \frac{A'P}{BP} \cdot \frac{BP}{PP'} - AO \cdot OD \text{ (and by similar triangles)}$$

$$= 2AI^2 \cdot \frac{IK}{AI} \cdot \frac{IK}{AI} - AO \cdot OD = 2r^2 - AO \cdot OD.$$

Again, from the similar triangles AIK , $P'PB$, we have

$$PP' \cdot IK \text{ or } 2Rr = AI \cdot PB = AI \cdot IP = SA^2 - SI^2,$$

therefore

$$SI^2 = SA^2 - 2Rr = R^2 - 2Rr.$$

Also, if AD be produced to meet the circumscribed circle in Q ,

$$2AO \cdot OD = AO \cdot OQ = SA^2 - SO^2,$$

therefore

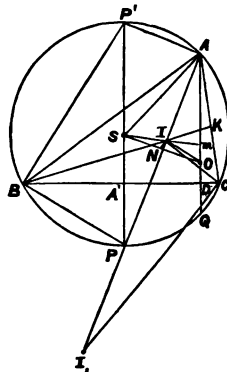
$$SO^2 = R^2 - 2AO \cdot OD.$$

Finally, if N be the centre of the nine-point circle; then, since it is the middle point of SO , we have

$$4NI^2 = 2SI^2 + 2IO^2 - SO^2$$

$$= 2R^2 - 4Rr + 4r^2 - 2AO \cdot OD - R^2 + 2AO \cdot OD = (R - 2r)^2,$$

hence $2NI = R - 2r$; and since R is twice the radius of the nine-point circle, this proves the theorem.



1. The average area of the triangle is

$$\begin{aligned}
 & \frac{12 \cdot 3^3}{4^3 \pi^3 r^3} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{2\pi} u \, d\theta \operatorname{cosec}^4 \phi \, d\phi \sin^2 \psi \sin^2 (\phi - \psi) \, d\psi \, r \\
 & \quad \times \sin \mu \, d\mu \cdot 2\pi x \, dx \cdot 64y^5 \, dy \sin \nu \, d\nu \, d\omega \\
 &= \frac{5184}{\pi r^3} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} d\theta \operatorname{cosec}^5 \phi \, d\phi \sin^3 \psi \sin^3 (\phi - \psi) \, d\psi \\
 & \quad \times \sin \mu \, d\mu x \, dx y^7 \, dy \\
 &= \frac{324r^2}{6\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \cos^3 \theta \, d\theta \operatorname{cosec}^5 \phi \, d\phi \sin^3 \psi \sin^3 (\phi - \psi) \, d\psi \sin^{11} \mu \, d\mu \\
 &= \frac{9216r^2}{385\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \cos^3 \theta \, d\theta \operatorname{cosec}^5 \phi \, d\phi \sin^3 \psi \sin^3 (\phi - \psi) \, d\psi \\
 &= \frac{192r^2}{385\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} (6\phi \cos \phi \operatorname{cosec}^3 \phi - 15\phi \cos \phi \operatorname{cosec}^5 \phi + 15 \operatorname{cosec}^4 \phi \\
 & \quad - 11 \operatorname{cosec}^2 \phi) \cos^3 \theta \, d\theta \, d\phi \\
 &= \frac{48r^2}{385\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \left\{ \left(\frac{1}{2}\pi + \theta \right) (15 - 12 \cos^2 \theta) + 15 \sin \theta \cos \theta - 2 \sin \theta \cos^3 \theta \right\} \\
 & \quad \times \cos^4 \theta \, d\theta = \frac{1}{7} \pi r^2.
 \end{aligned}$$

2. The average area of the inscribed circle of the triangle is

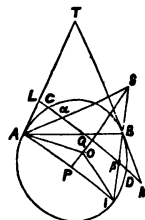
$$\begin{aligned}
 & \frac{12 \cdot 3^3}{4^3 \pi^3 r^3} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{2\pi} \pi x^2 \, d\theta \operatorname{cosec}^4 \phi \, d\phi \sin^2 \psi \sin^2 (\phi - \psi) \, d\psi \\
 & \quad \times r \sin \mu \, d\mu \cdot 2\pi x \, dx \cdot 64y^5 \, dy \sin \nu \, d\nu \, d\omega \\
 &= \frac{10368}{r^3} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} d\theta \operatorname{cosec}^4 \phi \, d\phi \operatorname{cosec}^2 \frac{1}{2} \phi \, d\phi \sin^2 \psi \sin^2 \frac{1}{2} \psi \\
 & \quad \times \sin^2 (\phi - \psi) \sin^2 \frac{1}{2} (\phi - \psi) \, d\psi \sin \mu \, d\mu x \, dx y^7 \, dy \\
 &= \frac{648r^2}{5} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \cos^3 \theta \, d\theta \operatorname{cosec}^4 \phi \, d\phi \operatorname{cosec}^2 \frac{1}{2} \phi \, d\phi \sin^2 \psi \sin^2 \frac{1}{2} \psi \\
 & \quad \times \sin^2 (\phi - \psi) \sin^2 \frac{1}{2} (\phi - \psi) \, d\psi \sin^{11} \mu \, d\mu \\
 &= \frac{18432r^2}{385} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} \int_0^{\frac{1}{2}\pi} \cos^3 \theta \, d\theta \operatorname{cosec}^4 \phi \, d\phi \operatorname{cosec}^2 \frac{1}{2} \phi \, d\phi \sin^2 \psi \sin^2 \frac{1}{2} \psi \\
 & \quad \times \sin^2 (\phi - \psi) \sin^2 \frac{1}{2} (\phi - \psi) \, d\psi \\
 &= \frac{96r^2}{1925} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{+\frac{1}{2}\pi+\theta} (60\phi + 30\phi \cos 2\phi + 15\phi \cos \phi \cos 2\phi - 143 \sin \phi \\
 & \quad + 38 \sin \phi \cos \phi + 99 \sin^3 \phi) \cos^3 \theta \, d\theta \operatorname{cosec}^4 \phi \, d\phi \operatorname{cosec}^2 \frac{1}{2} \phi \, d\phi \\
 &= \frac{96r^2}{1925} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \left\{ \left(\frac{1}{2}\pi + \theta \right) (42 \sin \theta - 14 \sin \theta \cos^2 \theta + 32 \sin \theta \cos^4 \theta - 42 + 60 \cos^2 \theta) \right. \\
 & \quad + 42 \cos \theta - 92 \cos^3 \theta - 42 \sin \theta \cos \theta - 32 \sin \theta \cos^3 \theta \\
 & \quad \left. + 45 \frac{1}{15} \cos^5 \theta + 32 \cos^5 \theta \log \left[\frac{1}{2} (1 - \sin \theta) \right] \right\} \cos^3 \theta \, d\theta \\
 &= \pi r^2 \left(\frac{3}{2} \frac{1}{5} - \frac{1}{2} \frac{1}{5} \log 2 \right).
 \end{aligned}$$

6917. (By J. L. MCKENZIE, B.A.)—A straight line XY cuts the tangents at two given points A and B on a given circle in L and M respec-

tively, and the circle itself in C and D; then (1) if the circles ABL, ABM cut XY in α and β respectively, prove $C\alpha = D\beta$; and (2) if the circles passing through L and M, and touching the given circle in B and A respectively, cut XY in α' and β' , prove $C\alpha' = D\beta'$.

Solution by Professor HADAMARD; BELLE EASTON; and others.

1. Prolongeons B β jusqu'à sa rencontre en I avec la circonférence et joignons AI, l'angle AIB est égal à l'angle LAB et par conséquent (puisque le quadrilatère LAB est inscriptible) à l'angle $\alpha\beta B$, ce qui montre que les droites $\beta\alpha$, IA sont parallèles. Mais alors les arcs CA, DI, ainsi que les cordes qui les sous-tendent, sont égaux, ainsi que les angles ACD, CDI. De plus l'angle $S\alpha\beta$, supplément de $A\alpha\beta$, est égal à ABI, supplément de ABM, puisque dans le quadrilatère inscrit AaBM les angles AaM, ABM, inscrits dans le même segment de cercle, sont égaux. Il est donc égal à $S\beta\alpha$, puisque $S\beta\alpha = TAB = TBA$. Les triangles CAa, DI β sont donc égaux comme ayant un côté égal (CA=DI) et deux angles égaux ($\angle ACD = \angle IDC$, $\alpha = \beta$). Donc $C\alpha = D\beta$.



2. Le point L [the modified figure can be easily imagined] appartenant à la tangente commune aux cercles ABC, Ma β a même puissance par rapport à tous deux. On en tire

$$La' = \frac{LC \cdot LD}{LM} \quad \text{et} \quad Ca' = \frac{LC(LM - LD)}{LM} = \frac{LC \cdot MD}{LM},$$

expression dont la symétrie montre que $Ca' = D\beta'$.

7263. (By J. J. WALKER, M.A.)—Show that the principal axes through any point (O) of a homogeneous solid, with respect to which the bounding surface (S) is convex, are also principal axes for the shell (S) supposed to have at each point a surface density proportional to the perpendicular from (O) on the tangent plane at that point.

Solution by the Rev. T. R. TERRY, M.A.; Professor MATZ, M.A.; and others.

Draw radii vectores from O to each point P of the shell S, and in OP take a point Q so that $\frac{OQ}{OP} = \lambda$, and over the surface S' formed by Q distribute uniformly a mass equal to that distributed uniformly over S. Then it is obvious that the products and moments of inertia round any axes through O for the shells S' and S are in ratio $\lambda^3 : 1$.

Now split up the solid into an infinite number of shells like S', and it follows at once that the products and moments of inertia round any axes through O for the solid are respectively one-third the corresponding products and moments for the shell S whose mass is the same as that of the solid and whose surface density at any point is proportional to the perpendicular from O on the tangent plane at that point.

7200 & 7233. (By G. S. CARR, B.A.)—(7200) If two non-intersecting circles, having radii a, b , centres A, B, and distance between the centres c , be inverted into concentric circles having their common centre at a distance d from A in the direction AB; show that α, β , the radii of the images, must have the values

$$\alpha = \frac{(2cd - a^2 + b^2 - c^2 + S)a}{a^2 - b^2 + c^2 - S}; \quad \beta = \pm \frac{(2cd - a^2 + b^2 - c^2 + S)b}{a^2 - b^2 - c^2 - S},$$

where

$$S = \sqrt{(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)},$$

and the + or - sign of β is to be read according as the smallest circle, radius b , lies within or without the other.

(7233.) If two non-intersecting circles, having radii a, b , centres A, B, and distance between the centres c , be inverted into concentric circles with radii α, β , and centre O; show that, if we denote

$$(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)^{\frac{1}{2}}$$

by S, we have

$$AO = \frac{a^2 - b^2 - c^2 - S}{2c} + \frac{k^2c}{S}, \quad \alpha = \frac{2k^2c^2a}{S(a^2 - b^2 + c^2 - S)}, \quad \beta = \frac{\pm 2k^2c^2b}{S(a^2 - b^2 - c^2 - S)},$$

a being $> b$ and the + or - sign being read according as the smallest circle is within or without the largest.

Solution by the Rev. T. R. TERRY, M.A.; Prof. NASH, M.A.; and others.

If C be the centre of inversion and $CA = p$, $CB = q$, we have

$$\frac{pk^2}{p^2 - a^2} = \frac{qk^2}{q^2 - b^2} = p - d, \quad \alpha = \frac{ak^2}{p^2 - a^2}, \quad \beta = \frac{bk^2}{q^2 - b^2}, \quad p - q = c \dots (1, 2, 3, 4).$$

Substituting for (4) in (1), we have

$$cp^3 - p(a^2 - b^2 + c^2) + a^2c = 0, \quad \therefore p = \frac{a^2 - b^2 + c^2 - S}{2c}, \quad \therefore q = \frac{a^2 - b^2 - c^2 - S}{2c},$$

which gives the values α and β in Quest. 7233. Also

$$AO = p - \frac{pk^2}{p^2 - a^2} = \frac{a^2 - b^2 + c^2 - S}{2c} + \frac{k^2c}{S}.$$

Again, from (1), substituting values of p and $p^2 - a^2$, we get

$$k^2 = \frac{(2cd - a^2 + b^2 - c^2 + S)S}{2c^2},$$

which, by substitution, gives the values of α and β Quest. 7200.

6879. (By J. O'REGAN.)—The horizontal base AB of a triangle being given in magnitude and position, and the side AC in magnitude only, construct the triangle so that a body may fall along CB from C to B in the least time possible.

Solution by Dr. CURTIS; G. M. REEVES, M.A.; and others.

This is a particular case of the following well-known problem—To find the line of quickest, or slowest, descent from a circle to a point, or *vice versa*, both being situated in the same vertical plane. Let B (Fig. 1) be the given point, and A and a the centre and radius of the given circle. Consider the time, t , of falling down any distance BC; produce BC to cut the circle in D; draw DE vertically and BE horizontally; denote BC, BD, DE by s , l , h , respectively; then $t^2 = \frac{2ls}{gh}$, but $ls (= AB^2 - a^2)$

is known; and therefore, when t is minimum or maximum, h is maximum or minimum, therefore D is one of the extremities of the vertical diameter. If the point B lie between the two horizontal tangents, there are two solutions for t minimum, the horizontal line through B corresponding to maximum time, ∞ ; when the point B lies on the same side of both horizontal tangents, one solution gives t maximum, the other minimum.

In the case proposed, denoting by α and β the angles BAC (Fig. 2) and ACD or ADC, we have

$$\frac{\pi}{2} + \alpha = 2\beta, \text{ and } \therefore \alpha = 2 \tan^{-1} \left(\frac{BA}{AD} \right) - \frac{\pi}{2}.$$

It can similarly be proved that, two circles being situated in the same vertical plane, the lines of quickest, or slowest, descent from one to the other will be the intercepts on the lines joining the extremities of the two vertical diameters of the circles.

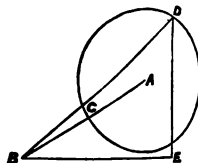


Fig. 1.

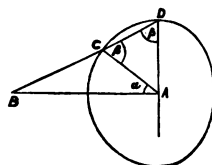


Fig. 2.

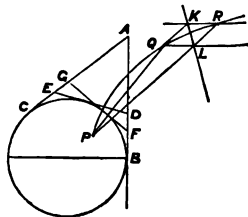
6817. (By T. C. SIMMONS, M.A.)—From Question 6720 deduce, by reciprocation, a property of the parabola, and give an independent proof of the theorem thus obtained.

Solution by the PROPOSER; E. RUTTER; and others.

Reciprocate the circle into a parabola whose focus is B (of course B being the centre of reciprocation).

Let P, Q, R correspond with lines AC, DE, FG; then RK, parallel to axis of parabola, will be the polar of F, and QL the polar of D. Also the chord PQ will be polar of E, and PR of G.

Hence K, the point of intersection of PQ and RK, will be the pole of EF, and similarly L the pole of DG; therefore KL is the polar of the intersection of DG and EF. But this point of intersection



lies on BC, therefore KL is perpendicular to BC. But the tangent at P is perpendicular to BC, therefore KL is parallel to the tangent at P.

[Hence we obtain the following theorem:—PQ, PR are any two chords of a parabola. If PQ meet the diameter through R in K, and PR meet the diameter through Q in L, then KL is parallel to the tangent at P].

6865. (By G. F. WALKER, M.A.)—Two equal particles are connected by a string which passes over a smooth peg; prove that, if one move vertically, and θ be measured from the vertical downwards, and u be the reciprocal of the radius vector, the equation of motion of the other is

$$\frac{\sin \theta}{u^3} + \frac{d}{d\theta} \left\{ \frac{\sin^2 \frac{1}{2} \theta + \frac{\sin \theta}{u} \frac{du}{d\theta}}{u^2 \left(2 \frac{d^2 u}{d\theta^2} + u \right)} \right\} = 0.$$

Solution by G. M. REEVES, M.A.; G. HEFFEL, M.A.; and others.

Measuring θ upwards, instead of downwards as in question, and taking mass of each particle to be unity, the equations of motion are

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = g \cos \theta - T, \quad \frac{1}{r} \cdot \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = -g \sin \theta,$$

and
$$\frac{d^2 r'}{dt^2} = g - T,$$

where T is tension of the string. Eliminating T , and observing that $r + r'$ is constant, we get

$$2 \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = -g (1 - \cos \theta) = -2g \sin^2 \frac{1}{2} \theta,$$

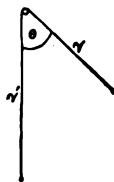
and
$$\frac{1}{r} \cdot \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = -g \sin \theta.$$

Let $r = \frac{1}{u}$, and $h = r^2 \frac{d\theta}{dt}$, so that $\frac{d\theta}{dt} = hu^2$. These equations become

$$2 \frac{dh}{dt} \cdot \frac{du}{d\theta} + 2h^2 u^2 \frac{d^2 u}{d\theta^2} + h^2 u^2 = 2g \sin^2 \frac{1}{2} \theta, \text{ and } u \frac{dh}{dt} = -g \sin \theta,$$

whence
$$h^2 = 2g \frac{\sin^2 \frac{1}{2} \theta + \frac{\sin \theta}{u} \frac{du}{d\theta}}{u^2 \left(2 \frac{d^2 u}{d\theta^2} + u \right)},$$

and
$$\frac{\sin \theta}{u^3} + \frac{d}{d\theta} \left\{ \frac{\sin^2 \frac{1}{2} \theta + \frac{\sin \theta}{u} \frac{du}{d\theta}}{u^2 \left(2 \frac{d^2 u}{d\theta^2} + u \right)} \right\} = 0.$$



$x^2 + y^2 + z^2 = 1$, satisfy the conditions

$$\Sigma(x) = 0 = \Sigma(y) = \Sigma(z), \quad \Sigma(yz) = \Sigma(zx) = \Sigma(xy) = 0,$$

$$\Sigma(x^2) = \Sigma(y^2) = \Sigma(z^2) = \frac{1}{3};$$

and writing $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ for (xyz) , we shall have the corners of a maximum tetrahedron inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, satisfying the

$$\text{conditions } \Sigma(x) = \Sigma(y) = \Sigma(z) = 0, \quad \Sigma(yz) = \Sigma(zx) = \Sigma(xy) = 0,$$

$$\Sigma(x^2) = \frac{1}{3}a^2, \quad \Sigma(y^2) = \frac{1}{3}b^2, \quad \Sigma(z^2) = \frac{1}{3}c^2,$$

and making $a^2 = \frac{1}{3}(B+C-A)$, $b^2 = \frac{1}{3}(C+A-B)$, $c^2 = \frac{1}{3}(A+B-C)$, we get the conditions required for kinetic equivalence to be as stated.

7218. (By Prof. GENESE, M.A.)—Prove that

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_2^2 & l \\ 0 & y & 0 & y_1^2 & y_2^2 & m \\ 0 & 0 & z & z_1^2 & z_2^2 & n \\ 0 & z & y & 2y_1z_1 & 2y_2z_2 & 2p \\ z & 0 & x & 2x_1x_1 & 2x_2x_2 & 2q \\ y & x & 0 & 2x_1y_1 & 2x_2y_2 & 2r \end{vmatrix} = -2\Delta \{ lX_1X_2 + \dots + p(Y_1Z_2 + Z_1Y_2) + \dots \}$$

where $\Delta = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$, X_1 is the minor of x_1 in Δ , &c.

7217. (By T. MUIR, M.A., F.R.S.E.)—Prove the identity

$$\begin{vmatrix} x & 0 & 0 & x_1^2 & x_1x_2 & x_2^2 \\ 0 & y & 0 & y_1^2 & y_1y_2 & y_2^2 \\ 0 & 0 & z & z_1^2 & z_1z_2 & z_2^2 \\ 0 & z & y & 2y_1z_1 & y_1z_2 + y_2z_1 & 2y_2z_2 \\ z & 0 & x & 2x_1x_1 & x_1x_2 + x_2x_1 & 2x_2x_2 \\ y & x & 0 & 2x_1y_1 & x_1y_2 + x_2y_1 & 2x_2y_2 \end{vmatrix} = |xy_1z_2|^3.$$

Solution by G. HEFFEL, M.A.; Professor ROY, M.A.; and others.

(7218). By suitably interchanging rows and columns, it may be shown that l, m, n , and p, q, r , are symmetrically involved with x, y, z .

$$\text{Coeff. of } l = z \begin{vmatrix} 0 & x & 2x_1y_1 & 2x_2y_2 \\ 0 & y & y_1^2 & y_2^2 \\ z & 0 & z_1^2 & z_2^2 \\ y & z & 2y_1z_1 & 2y_2z_2 \end{vmatrix} - y \begin{vmatrix} 0 & y & 2x_1z_1 & 2x_2z_2 \\ 0 & z & z_1^2 & z_2^2 \\ y & 0 & y_1^2 & y_2^2 \\ z & y & 2y_1z_1 & 2y_2z_2 \end{vmatrix}$$

$$\text{First term} = x^2 \begin{vmatrix} x & 2x_1y_1 & 2x_2y_2 \\ y & y_1^2 & y_2^2 \\ z & 2y_1z_1 & 2y_2z_2 \end{vmatrix} - yz \begin{vmatrix} x & 2x_1y_1 & 2x_2y_2 \\ y & y_1^2 & y_2^2 \\ 0 & z_1^2 & z_2^2 \end{vmatrix}$$

$$= 2y_1y_2z^2\Delta - xyz \begin{vmatrix} y_1^2 & y_2^2 \\ z_1^2 & z_2^2 \end{vmatrix} + 2x^2y \begin{vmatrix} y_1z_1 & y_2z_2 \\ x_1y_1 & x_2y_2 \end{vmatrix} - 2y^2z \begin{vmatrix} z_1^2 & z_2^2 \\ x_1y_1 & x_2y_2 \end{vmatrix}$$

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7228. (By Professor HUDSON, M.A.)—From a point A on the outer of two concentric circles, tangents AP , AQ are drawn to the inner; AP , QP meet the outer again in T , R ; prove that $RP : RQ = RT^2 : RA^2$.

Solution by T. A. E. SANDERSON; J. S. JENKINS; and others.

We have

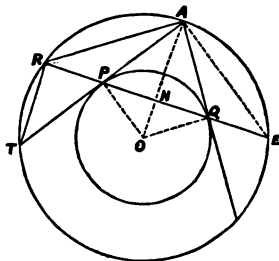
$$\angle ARP = \angle AER = \angle RTP;$$

hence ARP and ART are similar triangles, therefore

$$\frac{RT}{RA} = \frac{RP}{PA}; \text{ also } \frac{RP}{PA} = \frac{PA}{RQ},$$

because $RP \cdot RQ = PA^2$,

$$\therefore \frac{RP}{RQ} = \frac{RP}{PA} \cdot \frac{PA}{RQ} = \frac{RT^2}{RA^2}.$$



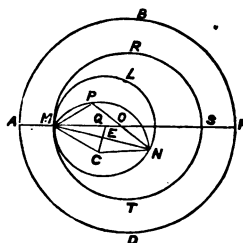
6894. (By Professor SEITZ, M.A.)—Three points are taken at random within a circle; prove that the mean value of the radius of the inscribed circle of the triangle formed by joining them is $\frac{192r}{35\pi^2} \left(\frac{128}{135} - \frac{\pi^2}{12} \right)$.

Solution by the PROPOSER.

Let $ABFD$ be the given circle, O its centre, and M, N, P the random points.

Through M describe the circle $MRST$ concentric with the given circle, and through N describe MLN tangent to the circle $MRST$, Q being its centre. Describe the arc MPN , C being its centre. Draw CQ , intersecting MN in E . Now, while M ranges over the surface of the given circle, let N be confined to the circle $MRST$, and P to the circle MLN . The whole number of relative positions of the three points, N and P being thus confined, is $\frac{1}{3}$ of the whole number of relative positions, if the points be taken indiscriminately over the surface of the given circle; and the triangle will evidently pass through all possible variations.

Let $OA = r$, $OM = x$, $MN = 2y$, $\angle OMN = \theta$, $\angle MCE = \phi$, $\angle PMN = \psi$, and z = the radius of the inscribed circle of the triangle MNP . Then we have $MP = 2y \sin(\phi - \psi) \operatorname{cosec} \phi$, $z = 2y \operatorname{cosec} \frac{1}{2} \phi \sin \frac{1}{2} \psi \sin \frac{1}{2}(\phi - \psi)$; an element of the circle at M is $2\pi x dx$, at N it is $4y dy d\theta$, and at P it is $d \cdot (MP) \sin(\phi - \psi) \cdot 2y \operatorname{cosec} \phi d\psi = 4y^2 \operatorname{cosec}^3 \phi \sin \psi \sin(\phi - \psi) d\phi d\psi$.



The limits of θ are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ϕ , 0 and $\frac{1}{2}\pi + \theta$, and doubled, since P may lie on either side of MN; of ψ , 0 and ϕ ; of x , 0 and r ; and of y , 0 and $x \cos \theta = y'$. Hence, since the whole number of ways the three points are taken is $\frac{1}{2}\pi^2 r^4$, the mean value of z is

$$\begin{aligned} & \frac{12}{\pi^2 r^5} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi+\theta} \int_0^\phi \int_0^r z \operatorname{cosec}^3 \phi \, d\theta \, d\phi \cdot 4 \sin \psi \sin(\phi - \psi) \, d\psi \cdot 2\pi x \, dx \cdot 4y^2 dy \\ &= \frac{768r}{35\pi^2} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi+\theta} \int_0^\phi \cos^5 \theta \, d\theta \operatorname{cosec}^3 \phi \operatorname{cosec} \frac{1}{2}\phi \sin \frac{1}{2}\psi \sin \frac{1}{2}(\phi - \psi) \sin \psi \\ &\quad \times \sin(\phi - \psi) \, d\psi \\ &= \frac{64r}{35\pi^2} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi+\theta} (3\phi \cos \phi \cos \frac{1}{2}\phi - 6 \sin \frac{1}{2}\phi + 14 \sin^3 \frac{1}{2}\phi) \cos^5 \theta \, d\theta \\ &\quad \times \operatorname{cosec}^3 \phi \operatorname{cosec} \frac{1}{2}\phi \, d\phi \\ &= \frac{64r}{105\pi^2} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \left\{ 3 \left(\frac{1}{2}\pi + \theta \right) (\sin^3 \theta - 1) + 9 \cos \theta + 3 \sin \theta \cos \theta - 5 \cos^3 \theta \right. \\ &\quad \left. - 3 \cos^3 \theta \log \left(\frac{1 - \sin \theta}{2} \right) \right\} \cos^5 \theta \, d\theta \\ &= \frac{192r}{35\pi^2} \left(\frac{128}{135} - \frac{\pi^2}{12} \right). \end{aligned}$$

7071. (By T. C. SIMMONS, M.A.)—Prove the two following constructions for finding the vertical chord of curvature at any point P of a parabola; (1) on PA produced take AR equal to three times the projection of the abscissa of P on AP; (2) from the focal distance SP cut off SV = AS, and on PS produced take SW = 3PS: draw a circle through the points A, V, W, meeting PA produced in R. In each of these cases, PR will be the required chord of curvature.

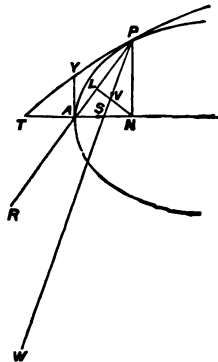
Solution by the PROPOSER; D. EASTWOOD, M.A.; and others.

1. Draw NL perpendicular to AP; and tangent PYT. Let PR be vertical chord of curvature. Then (as in the published Solutions to BESANT'S *Geometrical Conics*) we can show that

$$\begin{aligned} \text{PA} \cdot \text{PR} &= 4\text{PY}^2 = \text{PT}^2, \\ \text{therefore } \text{PA} \cdot \text{AR} + \text{PA}^2 &= \text{PN}^2 + \text{NT}^2, \\ \text{PA} \cdot \text{AR} + \text{PN}^2 + \text{AN}^2 &= \text{PN}^2 + 4\text{AN}^2, \\ \text{PA} \cdot \text{AR} &= 3\text{AN}^2 = 3\text{PA} \cdot \text{AL}, \\ \text{therefore } \text{AR} &= 3\text{AL}. \end{aligned}$$

$$\begin{aligned} 2. \quad \text{PA} \cdot \text{PR} &= 4\text{TY}^2 = 4\text{ST} \cdot \text{TA} \\ &= 4\text{SP} \cdot \text{PV} \\ (\text{since, if } \text{SV} = \text{AS}, \text{PV} &= \text{AT}) \\ &= \text{PW} \cdot \text{PV}; \end{aligned}$$

therefore a circle passes through the points V, W, A, and R.



6883. (By T. MORLEY, L.C.P.)—If the water-section of a ship be taken for the base of a right cylinder equal to the volume of the ship, and H be the height of the cylinder; prove that the vertical oscillations of the vessel are the same as the vibrations of a simple pendulum whose length is SH ; S being the specific gravity of ship and cargo.

Solution by Dr. CURTIS; G. M. REEVES, M.A.; and others.

M being the mass of the cylinder, A its principal section, and ξ its depression, we have

$$M \frac{d^2 \xi}{dt^2} = -Ag\xi, \text{ or } SHA \frac{d^2 \xi}{dt^2} + Ag\xi = 0, \text{ or } \frac{d^2 \xi}{dt^2} + \frac{g}{SH} \xi = 0;$$

comparing this with the equation of motion of a simple pendulum

$$\frac{d^2 \xi}{dt^2} + \frac{g}{l} \xi = 0,$$

we get $l = SH$. It is not difficult to extend this problem to the case where the reservoir in which the ship is immersed is, or may be considered, a cylinder whose section, $A + B$, is so small that a depression of the ship causes an elevation of the water in the reservoir. The solution then is $l = \left(\frac{B}{A+B} \right) SH$.

6731. (By W. J. C. SHARP, M.A.)—If any number of particles attract each other with a force varying directly as their masses, and inversely as the cube of the distances between them; prove that

$$\sum m (x^2 + y^2 + z^2) = at^2 + bt + c,$$

whatever the origin and axes may be.

Solution by D. EDWARDS; Professor SEITZ, M.A.; and others.

Denoting by Q half the sum of the products, two and two, of the masses divided by the squares of their respective distances, the equations of motion for a particle (m) at x, y, z are

$$m \frac{d^2 x}{dt^2} = \frac{dQ}{dx}, \quad m \frac{d^2 y}{dt^2} = \frac{dQ}{dy}, \quad m \frac{d^2 z}{dt^2} = \frac{dQ}{dz},$$

then

$$\begin{aligned} \frac{1}{2} \sum m \frac{d^2 r^2}{dt^2} &= \sum m \left(x \frac{d^2 x}{dt^2} + y \frac{d^2 y}{dt^2} + z \frac{d^2 z}{dt^2} \right) + \sum m \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\} \\ &= \sum \left(x \frac{dQ}{dx} + y \frac{dQ}{dy} + z \frac{dQ}{dz} \right) - 2 \sum \int \frac{mm'}{f^3} df, \end{aligned}$$

if f be the distance between two particles m, m' . But, by the nature of

the function Q , $\sum \left(x \frac{dQ}{dx} + y \frac{dQ}{dy} + z \frac{dQ}{dz} \right) = - \sum \frac{mm'}{f^2}$.

Hence $\sum m \frac{d^2 r^2}{dt^2} = 2a$, and therefore, &c.

7037. (By the EDITOR.)—A particle moves on the surface of an ellipsoid under the action of an attractive force varying inversely as the cube of the distance from the plane containing the greatest and least axis; show that, by a proper arrangement of the initial circumstances, the path will be one of the circular sections of ellipsoid passing through the centre.

Solution by W. H. BLYTHE, M.A.; E. RUTTER; and others.

Take the equation to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the attraction on the particle μy^{-3} parallel to axis y , taking the usual notation, the equations of motion are

$$\frac{a^2}{x} \cdot \frac{d^2x}{dt^2} = \frac{c^2}{z} \cdot \frac{d^2z}{dt^2} = \frac{b^2}{y} \left(\frac{d^2y}{dt^2} + \mu y^{-3} \right) = \frac{R}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{\frac{1}{2}}},$$

the mass of the particle taken as unity.

If the particle move in a central circular section, we have

$$x \left(\frac{1}{b^2} - \frac{1}{a^2} \right)^{\frac{1}{2}} = \pm z \left(\frac{1}{c^2} - \frac{1}{b^2} \right)^{\frac{1}{2}};$$

it is evident that this equation cannot co-exist with the above unless $\frac{d^2x}{dt^2} = 0 = \frac{d^2y}{dt^2}$, and therefore $\frac{d^2y}{dt^2} + \mu y^{-3} = 0$. Solving these equations,

$$x = na \left(\frac{b^2 - c^2}{a^2 - c^2} \right)^{\frac{1}{2}} [\cos \theta + a_1], \quad z = nc \left(\frac{a^2 - b^2}{a^2 - c^2} \right)^{\frac{1}{2}} [\cos \theta + a_1], \quad y = nb \sin \theta,$$

where $\cos \theta = mt + p$, and $\mu = m^2 n^3 b^4$. Substituting these values in the equation to the ellipsoid, noticing that they must be true for all values of θ , we obtain $a_1 = 0$ and $n = 1$; and, taking

$$\left(\frac{ds}{dt} \right)^2 = \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 = b^2 \left(\frac{d\theta}{dt} \right)^2$$

from the above equations, therefore $\frac{ds}{d\theta} = b$; integrating from $\theta = 0$ to 2π , we obtain $s = 2b\pi$.

4387. (By Professor CAYLEY, F.R.S.)—Using the term "Cassinian" to denote a bi-circular quartic having four foci in a right line; show that the equation of a Cassinian having for its four foci the points $x=a$, $x=b$, $x=c$, $x=d$ on the axis of x , may be written in the four equivalent forms

$$\begin{vmatrix} \tau(a-c), & \sigma(b-d), & \rho(c-b) \\ \tau(c-d), & \cdot, & \rho(d-a), & \sigma(a-c) \\ \sigma(b-d), & \rho(a-d), & \cdot, & \tau(b-a) \\ \rho(b-c), & \sigma(c-a), & \tau(a-b), & \cdot \end{vmatrix} (A^{\frac{1}{2}}, B^{\frac{1}{2}}, C^{\frac{1}{2}}, D^{\frac{1}{2}}) = 0,$$

Solution by J. HAMMOND, M.A.; BELLE EASTON; and others.

Since $v_n = \frac{u_{n+1}}{u_n}$, we have $2u_{n+1}^2 = (u_n + u_{n+1}) \frac{u_{n+2}}{u_{n+1}}$ (1).

In (1) put $u_n = u_{n+1} w_n$, then we have

$$2u_{n+2}^2 w_{n+1}^2 = (u_{n+1} w_n + u_{n+1}) \frac{u_{n+2}}{u_{n+1}}, \text{ or } 2w_{n+1}^2 = w_n + 1 \text{ (2).}$$

The solution of (2) is $w_n = \cos \left(\frac{\alpha}{2^n} \right)$, where α is an arbitrary periodic constant; hence

$$u_{n+1} = u_n \sec \left(\frac{\alpha}{2^n} \right), \text{ or } u_n = C \sec \left(\frac{\alpha}{2^{n-1}} \right) \sec \left(\frac{\alpha}{2^{n-2}} \right) \text{};$$

and $v_n = \sec \left(\frac{\alpha}{2^n} \right) u_{n+1}$, or $v_n = C \sec^2 \left(\frac{\alpha}{2^n} \right) \sec \left(\frac{\alpha}{2^{n-1}} \right) \sec \left(\frac{\alpha}{2^{n-2}} \right) \dots$

From these given may be deduced the somewhat simpler solutions

$$u_n = 2^n \sin \left(\frac{\beta}{2^n} \right) A, \quad v_n = 2^{n+1} \tan \left(\frac{\beta}{2^{n+1}} \right) A.$$

6197. (By Professor MOREL.)—Résoudre $x^{\frac{1}{2}} - x^{\frac{1}{3}} - 7x^{\frac{1}{4}} + x^{\frac{1}{5}} + 6 = 0$.

Solution by J. HAMMOND, M.A.; J. O'REGAN; and others.

Putting $x = y^6$, we have

$$y^4 - y^3 - 7y^2 + y + 6 = (y^2 - 1)(y^2 - y - 6) = 0;$$

whence $y = -1, 1, -2, 3$, and $x = 1, 1, 2^6, 3^6$.

5618. (By Professor SEITZ, M.A.)—The perimeter of an inscriptible quadrilateral is $2a$; show that (1) its average area is $\frac{1}{2}\pi a^2 \left(\frac{e^{\frac{1}{2}}}{2} - \frac{1}{2}\pi \right)$; (2) if the quadrilateral is also circumscribable, its average area is $\frac{1}{8}\pi^2 a^2$; and (3) the average area of the inscribed circle is $\frac{1}{8}\pi a^2$.

Solution by the PROPOSER.

1. Let ABCD be the quadrilateral, and put AB = x , BC = y , CD = z , DA = $2a - x - y - z$, area ABCD = u ; then

$$u = [(a-x)(a-y)(a-z)(x+y+z-a)]^{\frac{1}{2}};$$

the limits of x are 0 and a ; from $y = 0$ to $y = a - x = s$ the limits of z are $a - x - y = t$ and a ; and from $y = s$ to $y = a$ the limits of z are 0 and

$$\text{and} \quad \frac{\rho(b-c)a}{(r+a)^{\frac{1}{2}}} + \frac{\sigma(c-a)\beta}{(r+\beta)^{\frac{1}{2}}} + \frac{\tau(a-b)\gamma}{(r+\gamma)^{\frac{1}{2}}} = 0,$$

$$\text{or} \quad \frac{\rho^2(b-c)^2}{\beta-\gamma} + \frac{\sigma^2(c-a)^2}{\gamma-a} + \frac{\tau^2(a-b)^2}{a-\beta} = 0,$$

$$\text{or} \quad \frac{\rho^2(b-c)}{b+c-2x} + \frac{\sigma^2(c-a)}{c+a-2x} + \frac{\tau^2(a-b)}{a+b-2x} = 0,$$

and, by the equation of condition,

$$\rho^2(b-c) \left\{ \frac{e-d}{b+c-2x} - \frac{a-d}{a+b-2x} \right\} + \sigma^2(c-a) \left\{ \frac{e-d}{c+a-2x} - \frac{b-d}{a+b-2x} \right\} = 0,$$

$$\text{or} \quad \frac{(b+d-2x)(c+a-2x)}{\sigma^2} = \frac{(a+d-2x)(b+c-2x)}{\rho^2},$$

which, by symmetry, proves the given form of the equation.

7269. (By **BELLE EASTON.**)—One of two bags contains 10 sovereigns, and the other 10 shillings. One coin is taken out of each and placed in the other. This is repeated ten times. Find now the expectation of each bag.

Solution by the Rev. T. R. TERRY, M.A.; J. O'REGAN; and others.

Let u_n be the value in shillings of the sovereign bag after n operations; then $u_{n+1} = \frac{1}{10}u_n + \frac{1}{10}(210 - u_n) = \frac{1}{2}u_n + 21$, therefore $u_n = A(\frac{1}{2})^n + 105$. But, when $n = 0$, $u_n = 200$, therefore $A = 95$; therefore the values of the bags after n operations are

$$105 + 95 \cdot (\frac{1}{2})^n, \text{ and } 105 - 95 \cdot (\frac{1}{2})^n \text{ respectively.}$$

4897. (By the late Professor **CLIFFORD**, F.R.S.)—Let $U, V=0$ be the point-equations, and $u, v=0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u \pm \lambda v = 0$, they will intersect on $U - \lambda^2 V = 0$. This last conic passes through the points of contact of the conics $u \pm \lambda v = 0$ with the common tangents of u and v .

Solution by W. J. C. SHARP, M.A.

Referring the conics to their common self-conjugate triangle, let

$$U \equiv ax^2 + by^2 + cz^2, \quad V \equiv a'x^2 + b'y^2 + c'z^2,$$

then, if $A = bc$, &c., we have

$$u = Aa^2 + B\beta^2 + C\gamma^2, \text{ and } v = A'a^2 + B'\beta^2 + C'\gamma^2;$$

7283. (By Professor MARTIN, M.A., Ph.D.)—From a full cask of wine a quantity is drawn off at random, and the cask filled up with water. Find the chance that, after n such operations, the cask contains more wine than water.

Solution by the PROPOSER.

Let $x_1, x_2, x_3, \dots, x_n$ denote the quantities of the liquid taken out at the 1st, 2nd, 3rd, ..., n th drawings, a = the capacity of the cask, and let one- m th be the *part* of the contents the wine must exceed.

Then $a \left(1 - \frac{x_1}{a}\right) \left(1 - \frac{x_2}{a}\right) \left(1 - \frac{x_3}{a}\right) \dots \left(1 - \frac{x_n}{a}\right)$ = quantity of wine

remaining after the n th drawing, which must exceed $\frac{a}{m}$. Putting

$$a \left(1 - \frac{x_1}{a}\right) \left(1 - \frac{x_2}{a}\right) \left(1 - \frac{x_3}{a}\right) \dots \left(1 - \frac{x_n}{a}\right) = \frac{a}{m}, \text{ we have}$$

$$x_1 = a \left\{ 1 - \frac{1}{m \left(1 - \frac{x_2}{a}\right) \left(1 - \frac{x_3}{a}\right) \left(1 - \frac{x_4}{a}\right) \dots \left(1 - \frac{x_n}{a}\right)} \right\} = a_1.$$

The greater x_1 is, the less wine will be left; therefore x_1 may have any value from 0 to a_1 . Therefore

$$p_1 = \frac{a_1}{a} = 1 - \frac{1}{m \left(1 - \frac{x_2}{a}\right) \left(1 - \frac{x_3}{a}\right) \left(1 - \frac{x_4}{a}\right) \dots \left(1 - \frac{x_n}{a}\right)}$$

is the probability on the supposition that $x_2, x_3, x_4, \dots, x_n$ are known.

The greatest value x_r can have is when $x_1, x_2, x_3, x_4, \dots, x_{r-1}$ each = 0, and the limits of $x_r, x_2, x_3, \dots, x_n$, are 0 and (respectively)

$$a \left\{ 1 - \frac{1}{m \left(1 - \frac{x_{r+1}}{a}\right) \left(1 - \frac{x_{r+2}}{a}\right) \left(1 - \frac{x_{r+3}}{a}\right) \dots \left(1 - \frac{x_n}{a}\right)} \right\} = a_r,$$

$$a \left\{ 1 - \frac{1}{m \left(1 - \frac{x_2}{a}\right) \left(1 - \frac{x_4}{a}\right) \left(1 - \frac{x_6}{a}\right) \dots \left(1 - \frac{x_n}{a}\right)} \right\} = a_2,$$

$$\dots \dots \dots \dots \dots \dots$$

$$a \left(1 - \frac{1}{m}\right), = a_n.$$

$$\text{Hence probability is } p_n = \frac{\int_0^{a_n} \dots \int_0^{a_4} \int_0^{a_3} \int_0^{a_2} p_1 dx_n \dots dx_4 dx_3 dx_2}{\int_0^{a_n} \dots \int_0^{a_4} \int_0^{a_3} \int_0^{a_2} dx_n \dots dx_4 dx_3 dx_2}$$

$$= \frac{1}{a^{n-1}} \int_0^{a_n} \dots \int_0^{a_4} \int_0^{a_3} \int_0^{a_2} \left\{ 1 - \frac{1}{m \left(1 - \frac{x_2}{a}\right) \left(1 - \frac{x_3}{a}\right) \left(1 - \frac{x_4}{a}\right) \dots \left(1 - \frac{x_n}{a}\right)} \right\} \\ \times dx_n \dots dx_4 dx_3 dx_2.$$

7280. (By Professor MALET, F.R.S.)—If any three conjugate semi-diameters of the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = \frac{1}{k^2}, \quad \frac{x^2}{a''^2} + \frac{y^2}{b''^2} + \frac{z^2}{c''^2} = \frac{1}{k'^2},$$

meet the three surfaces respectively in the points P, Q, R, prove that the envelope of the plane PQR is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{1 + k^2 + k'^2}.$$

Solution by REV. T. R. TERRY, M.A.; J. P. JOHNSTONE; and others.

Take the three conjugate diameters as axes. Let the intercepts on these axes made by the first ellipsoid be A, B, C, by the second $\frac{A}{k}, \frac{B}{k}, \frac{C}{k}$, and by the third $\frac{A}{k'}, \frac{B}{k'}, \frac{C}{k'}$. Then the equation to the plane PQR is

$$\frac{x}{A} + \frac{ky}{B} + \frac{k'z}{C} = 1 \dots\dots\dots(1),$$

and the equation to a fourth ellipsoid similar to the other three, and having semi-axes $\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda}$, is

$$\frac{\lambda^2 x^2}{A^2} + \frac{\lambda^2 y^2}{B^2} + \frac{\lambda^2 z^2}{C^2} = 1 \dots\dots\dots(2).$$

But (1) will touch (2) if $\lambda^2 = 1 + k^2 + k'^2$, which proves the required result.

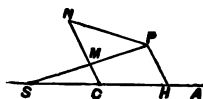
7299. (By R. KNOWLES.)—P is any point on an ellipse of which S and H are the foci; circles are drawn on SP and HP as diameters; show that (1) the loci of their centres are two equal ellipses intersecting on the minor axis of the given ellipse; (2) the circles touch the auxiliary circle; (3) the line joining the points of contact touches the ellipse at P.

Solution by ARTHUR COHEN, Q.C., M.P.; C. BICKERDIKE; and others.

1. Draw CMN parallel to HP, meeting SP in M, and the auxiliary circle in N, so that CN = CA the semi-major axis; then CM = $\frac{1}{2}$ HP, SM = $\frac{1}{2}$ SP, therefore

$$CM + SM = \frac{1}{2}(SP + HP) = CA = CN,$$

therefore NM = SM = MP, and the circle on SP as diameter touches the auxiliary circle at N.



2. Since SM + CM = CA, the locus of M is an ellipse with S and C as foci and major axis = CA. Similarly the locus of the middle point of HP is an ellipse with H and C as foci and major axis = CA.

3. Let O be the point where the two equal ellipses meet. Then we have

$$SO + CO = CA = HO + CO, \text{ therefore } SO = HO,$$

therefore O is a point on the perpendicular bisector of SH or on the minor axis of the ellipse. Again, since $NM = MP$, NP makes equal angles with NM and MP, or with HP and PS, and is therefore the tangent at P. Hence the line joining the point of contact with the auxiliary circle is the tangent at P.

7277. (By Professor SYLVESTER, F.R.S.)—If S_k denotes the sum of ~~the~~ k -ary products of $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i-1}$, prove that the coefficient of ~~any~~ positive $(1+k)^{\text{th}}$ power of $\frac{1}{x}$, in the development of $\frac{1}{(1-e^{-x})^i}$ in a series of ascending powers of x , is $i! \cdot S_k$.

Solution by the PROPOSER.

Let $R_x, R_1 x$ in general denote certain series containing only positive powers of x , then

$$\begin{aligned} \frac{1}{(1-e^{-x})^i} &= \left(1 - \frac{1}{2} \frac{d}{dx}\right) \left(1 - \frac{1}{2} \frac{d}{dx}\right) \dots \left(1 - \frac{1}{i-1} \frac{d}{dx}\right) \left(\frac{1}{1-e^{-x}}\right) \\ &= \left\{1 - S_1 \frac{d}{dx} + S_2 \left(\frac{d}{dx}\right)^2 \dots\right\} \left\{\frac{1}{x} + R_x\right\} \\ &= \frac{1}{x} + \frac{S_1}{x^2} + \frac{1 \cdot 2 S_2}{x^3} + \frac{1 \cdot 2 \cdot 3 S_3}{x^4} + \dots + \frac{1 \cdot 2 \dots (i-1) S_{i-1}}{x^i} + R_1 x. \end{aligned}$$

Hence the theorem is proved.

It will be observed that the gist of the proof consists in separating the series for $\frac{1}{(1-e^{-x})^i}$ into i separate series, each containing a single negative power of x , the solitary and isolated sentinel as it were of an infinite host of terms.

7246. (By Capt. MACMAHON, R.A.)—From any point P in the side BC of the triangle ABC, lines PD, PE are drawn parallel to AC and AB, meeting those sides in D and E; prove that the envelope of DE is a parabola, whose focus is at a distance from A equal to $bc[2(b^2 + c^2) - a^2]^{-1/2}$.

Solution by Dr. CURTIS; C. MORGAN, M.A.; and others.

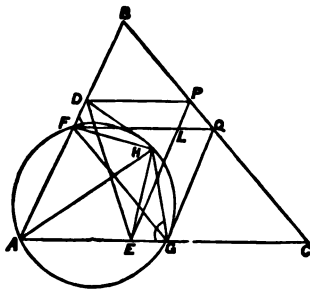
From Q, the mid-point of BC, draw QF, QG parallel respectively to AC, AB; through F, G, A draw a circle, and find on it the point H, such that FH : GH = FA : GA; then, drawing DH, EH, we have

$$DF : EG = PL : LQ = FA : GA \\ = FH : GH;$$

therefore $DF : FH = EG : GH$,
and $\angle DFH = \angle EGH$;

hence $\triangle DFH, \triangle EGH$ are similar triangles and $\angle FDH = \angle GEH$; therefore the known point, H, is on the circle circumscribing the triangle ADE; if therefore from the point H perpendiculars be drawn on the fixed lines AB, AC, and the variable line DE, the feet of these three perpendiculars will lie on the fixed straight line joining the two former. The envelope required is therefore the envelope of the straight line which is perpendicular to any variable line passing through a given point at the point in which this line intersects a given fixed line, viz., a parabola whose focus is at H, and whose tangent at vertex is the line which joins the feet of the perpendiculars let fall on AB, AC, from the point H. It is not difficult to prove that

$$AH = bc [2(b^2 + c^2) - a^2]^{-\frac{1}{2}}.$$



7232. (By R. F. SCOTT, M.A.)—If (a, b, c) stand for the straight line $ax + by + cz = 0$, show that the six lines obtained by permutating a, b, c in (a, b, c) in all possible ways, all touch the conic

$$\frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{2(yz + zx + xy)}{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab} = 0.$$

Solution by G. F. WALKER, M.A.; Professor MATZ, M.A.; and others.

The line $ax + by + cz = 0$ will touch the conic $Lx^2 + \dots + 2L'yz + \dots = 0$, if $a^2(MN - L'^2) \dots + 2bc(M'N' - LL') + \dots = 0$. This will be satisfied for all interchanges of a, b, c , if $L = M = N = P$ suppose, and $L' = M' = N' = Q$, that is, if

$$(a^2 + \dots)(P^2 - Q^2) + 2(bc + \dots)(Q^2 - PQ) = 0,$$

that is,

$$(a^2 + \dots)(P + Q) - 2(bc + \dots)Q = 0;$$

and the conic will be

$$\frac{x^2 + \dots}{a^2 + \dots} - \frac{2(yz + \dots)}{a^2 + \dots - 2bc - \dots} = 0.$$

where $h^2 + k^2 = p^2$, the centre of similitude (ξ, η) lies on the line

$$y = \frac{k}{h-b}(x-b), \text{ and } \xi-b : \xi-h = c : r ;$$

hence, by elimination, $c^2 p^2 = (c-r)^2 (\xi^2 + \eta^2) + 2r(c-r)\xi b + r^2 b^2$, a circle; and therefore the required envelope is a conic with a focus at the given focus.

Quests. 6964, 7081, 7120. Solution by the PROPOSER.*

(6964).—All that I have done, or am likely to do, towards the solution of this question, is to construct the 343 24-edra, which, having only triad summits, and no faces less than 5-gons nor greater than 8-gons, reduce by the rules to the solid 5^{12} . Those rules of construction and reduction, exact converses of each other, are clearly laid down in a short tract of mine, which our learned Mathematical Editor has kindly undertaken to supply for the asking to any of his contributors. It fills a few pages of the *Proceedings* (1882-3) of the *Literary and Philosophical Society of Liverpool*. This process of construction of n -edra on any $(n-c)$ edron as base is definite and exhaustive, with repetitions few and all foreseen.

The figures and labels of the 343 solids are before me; but it is not worth the while, even if space permitted, to give them here. Such detail may come to be of interest, but not until a few Professors have really walked into this subject,—a walk rather long and arduous, which will not come off before somebody's day.

Under the signatures of the 24 faces, there are, among these 343, under

$8^4 5^{20}$,	1;	$8^3 6^3 5^{18}$,	2;	$8^3 7 6 5^{19}$,	2;
$8^2 7^2 6^2 5^{18}$,	12;	$8^2 7 6^4 5^{17}$,	15;	$8^2 6^6 5^{16}$,	7;
$8 7^4 6 5^{18}$,	2;	$8 7^3 6^3 5^{17}$,	26;	$8 7^2 6^5 5^{16}$,	41;
$8 7 6^7 5^{15}$,	47;	$8 6^5 5^{14}$,	9;	$7^6 5^{18}$,	3;
$7^5 6^2 5^{17}$,	4;	$7^4 6^4 5^{16}$,	33;	$7^3 6^6 5^{15}$,	55;
$7^2 6^8 5^{14}$,	70;	$7 6^{10} 5^{12}$,	13;	$6^{12} 5^{12}$,	1.

Of these 343, 59 are symmetrical, having faces of which some are either repetitions or reflected images of others. Many species of polyedral symmetry are found in the 59, but by no means all. The number of different pentagons on the 343 solids is 4896, giving so many different smoking caps for the Sultan of Borneo, constructible on the fundamental 5^{12} . The number of 24-edra built on the fundamentals $6^2 5^{12}$, $7^2 5^{14}$, $8^2 5^{16}$, &c., may turn our hundreds into thousands. Hence the benevolent reader can have no sufficient excuse, at least for the next few milleniums, for writing to that potentate on this painful subject.

Of these 343 solids, 217 are either once only made, or twice made (or makeable on another fundamental). Only one is above four times makeable by the rules.

(7081 & 7120).—For the solution of MINERVA's riddle in Quest. 7081, we have to construct n -edra having only triad summits, and only 5-gonal and 6-gonal faces, on which every edge 55 is in at least one summit 555. I need not say that about any edge the dancers are the faces forming the edge, while the singers are the two faces which the edge meets at its two summits.

* The Questions, being long, are given in *Contents* only.

1. After the fundamental $6^{25}12$, the first solution, for five heroes, is constructed (not regularly) by placing a point on every edge 54 of a 5-sided prism, and crowning the two wallings $6, 6, 6, 6, 6$, with frames 5_6 . The solid formed is A_{17} , $6^{55}12$ a 5-zoned Monarchaxine Heterozone. The zones are $650^30^3, 50^3$; the two 5-zoned poles are 5-gons on the janal principal axis, and the five secondary heteroid axes carry each for poles a 2-zoned 6-gon and a 2-zoned edged 66.

2. If we treat either polar 5-gon of A_{17} as we treated the polar 5-gons of the prism, we get B_{22} , $6^{105}12$, a 5-zoned Monarchaxine Homozone, whose frames are as before, $5_6, 5_6$; a solution for ten heroes.

There are six janal axes, one principal and 5-zoned with polar 5-gon and five identical zones 860^30^3 , and five zoneless 2-ple secondary axes each carrying polar edges 6, all ten alike. The operation can be repeated on B_{22} , giving C_{27} , on which in like manner is built D_{32} , &c. Thus for every value of m we construct the Monarchaxine Heterozones $A_{17}, C_{27}, \dots E_{17} + 10m$, and the Monarchaxine Homozones $B_{22}, D_{32}, \dots F_{22} + 10m$; which have the frames $5_6, 5_6$, and are all solutions of our riddle, having at least one summit 555 on every edge 55.

3. Placing a point on every edge 65 of any 6-gon of A_{17} , we have the walling $6, 6, 6, 6, 6, 6$. On this we impose the frame 2_4 , and thus construct, properly by a least frame, G_{20} , $6^{85}12$, a 3-zoned Monarchaxine Heterozone. The frames are $2_4, 2_4, 2_4$, the zones are three 264^60 , about the janal principal axis whose poles are 3-zoned 6-gons, and a fourth secondary zone, 660^30^3 , met by three secondary two-zoned heteroid axes, whose zoned polar edges are on each 55 and 66. This G_{20} is a solution for eight heroes of this riddle of PALLAS, every edge 55 having a summit 555.

4. On alternate edges 64 of a six-sided prism, place six ray-points two and two, and in the opposite 6-gon do the like, making no 7-gon and leaving no 4-gon. Crowning the two wallings $6_6, 6_6, 6_6, 6_6$ with frames $4_{006}, 4_{006}$, we get H_{18} , $6^{65}12$, a Monarchaxine Homozone, having 3-zoned polar triaces on its principal janal axes, and on its three janal 2-ple secondaries six zoneless polar edges 66 all alike, which we write $(66)^2 = 6$. The zone is 460^40^3 . Every edge 55 is in a summit 555.

5. If we place our six points so as to make two contours 210210, instead of 202020, leaving no 4-gon, we can impose two frames 4_{006} ; and we thus complete I_{18} , $6^{65}12$, a zoneless Triaxine. There are three different 2-ple janal axes, whose six poles are two 2-ple 6-gons, two edges 66 and two edges 55. H_{18} and I_{18} are solutions for six heroes.

7. Remove a frame of H_{18} , and plant a 6-gon on every zero edge of the contour. The contour is still 202020, and the frame can be reimposed. This gives us a solution for nine heroes, J_{21} , $6^{95}12$, a 3-zoned Monarchaxine Heterozone, whose frames are $4_{006}, 4_{006}$, and zones 470^50^3 and 63^30^3 . Repeating this on J_{21} , we get K_{24} , $6^{125}12$, a 3-zoned Monarchaxine Homozone; 480^60^3 , $(66)^2 = 6$. Thus we can continue to augment the 6-gons by three without limit, retaining the frames $4_{006}, 4_{006}$, on our Monarchaxine Heterozones and Homozones.

8. Removing a frame of I_{18} , we can plant two 6-gons on the two zero edges of the contour, and replacing the frame, obtain for the case of eight heroes L_{20} , a zoneless Triaxine, having still the frames $4_{006}, 4_{006}$. And thus, retaining these frames, we can increase the 6-gons on these Triaxines by two without limit. All these results in (7) and (8) are solutions of PALLAS's riddle, for squads of heroes innumerable.

9. There is also a $6^{125}12$, a zoneless 3-ple Monarchaxine, whose two principal janal poles are zoneless 3-ple triaces 666, and whose three 2-ple heteroid axes, all alike, carry each zoneless polar edges 66 and 55. The

frames are 24, 24, 24. The student will easily construct this solution of our riddle for 12 heroes, if it is worth his while to try.

10. Inside a 6-gon P draw faces 656565, and three more 6-gons symmetrically collateral with those just drawn and with an eighth. There are three contours 101010. Crowning them, and also P, with 1_3 , 1_3 , 1_3 , 1_3 , so as to make no 7-gon, we get M_{22} , $6^{10}5^{12}$, $480'0''$, a zoned Tetrarchaxine. The four principal heteroid axes, all alike, have for poles a 3-zoned 6-gon and a triace 555; and six two-zoned 6-gons terminate the three janal secondary 2-zoned axes.

This is a solution for ten heroes, not only of MINERVA's riddle 7081, but also of APOLLO's in Quest. 7120; for every edge 55 is in one and only in one summit 555.

11. A second solution of Quest. 7120 is got by crowning every 6-gon of the zoned Tetrarchaxine 6^45^{12} by 1_3 , so as to make no 7-gon. The solid formed is a zoneless Tetrarchaxine $6^{12}5^{12}$, having seven zoneless axes, four 3-ple and three 2-ple, the latter being janal axes.

A third solution is obtained by crowning with 1_3 every principal polar 6-gon on M_{22} , so as to make no 7-gon. The solid formed is $6^{12}5^{12}$ a zoned Tetrarchaxine, having seven zoned axes.

12. The solid of URANIA in Quest. 7120 is the zoned Hexarchaxine $6^{30}5^{12}$ having six 5-zoned, ten 3-zoned, and fifteen 2-zoned axes. On every face of 5^{12} place a 5-sided prism; then efface the edges of 5^{12} . The reader easily sees,

“That every four, that glorious night,
Would be heroes three, and a goddess bright.”

With the help of the tract mentioned in the Solution of Quest. 6964, what is written above will be readily intelligible.

6947. (By G. F. MERRITT, M.A.)—The tangent and normal to an ellipse at the point P cut the axes in T and t, G and g, respectively; GT, gt are bisected in O and o; Oo cuts the normal Pg in Q: show that PQ is one-half of the radius of curvature at P.

Solution by R. KNOWLES, B.A.; Prof. HADAMARD; and others.

If P be $(x_1 y_1)$, we find

$$Ct = \frac{a^2}{x_1}; \quad CT = \frac{b^2}{y_1}; \quad Cg = e^2 x; \quad Cg = -\frac{a^2 e^2}{b^2} y_1; \quad CG = e^2 x_1;$$

$$OC = \frac{1}{2} \left(\frac{b^4 - a^2 e^2 y_1^2}{b^2 y_1} \right); \quad Co = \frac{1}{2} \left(\frac{a^2 + e^2 x_1^2}{x_1} \right);$$

$$\text{equation to} \quad Oo \quad \frac{2x_1}{a^2 + e^2 x_1} + \frac{2b^2 y_1}{b^4 - a^2 e^2 y_1^2} = 1 \dots \dots \dots (1);$$

and from this and equation to normal we obtain

$$x = \frac{x_1 (a^2 + e^2 x_1^2) (b^4 + a^2 e^2 y_1^2)}{2a^2 b^2 (a^2 - e^2 x_1^2)} = \frac{x_1 (a^2 + e^2 x_1^2)}{2a^2} \quad \text{and} \quad x - x_1 = \frac{x_1 (a^2 - e^2 x_1^2)}{2a^2}.$$

$$\begin{aligned} \text{Now} \quad PQ^2 &= (y - y_1)^2 + (x - x_1)^2 = \frac{a^4 y_1^2 + b^4 x_1^2}{b^4 x_1^2} (x - x_1)^2 \\ &= \frac{a^2 b^2 (a^2 - e^2 x_1^2)}{b^4 x_1^2} \cdot \frac{x_1^2 (a^2 - e^2 x_1^2)^2}{4a^4} = \frac{(a^2 - e^2 x_1^2)^3}{4a^2 b^2}; \end{aligned}$$

therefore PQ = $\frac{1}{2}$ radius of curvature at P.

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Solution by R. KNOWLES, B.A.; W. H. BLYTHE, M.A.; and others.

If the inscribed circle touch the side BC at D, and the escribed circle, which touches BC between B and C, touch it at D; then

$$2BD = AB + BC - AC = 2CD,$$

therefore $BD = CD$, and the mid-point of DD' is the same as that of BC , or the tangents from this point to the two circles are equal, and this point is on the radical axis of the circles, which is therefore the perpendicular from the middle point of BC on the line joining their centres, i.e., the internal bisector of the angle at A.

Similarly, if a perpendicular be drawn from the middle point of a side of a triangle upon the external bisector of the opposite angle; this is the radical axis of the two escribed circles whose centres lie on their bisector. Hence the radical axes are the bisectors, internal and external, of the angles of the triangle formed by joining the middle point of the sides, and the radical centres are the centres of the inscribed and escribed circles of that triangle.

5712. (By Professor MALET, F.R.S.)—Express in terms of Elliptic Functions the complete integral of the linear equation

$$\frac{d^2y}{dx^2} + \frac{3(1-2x)}{x(1-x)} \frac{dy}{dx} + \frac{1-7x+7x^2}{x(1-x)^2} \frac{dy}{dx} - \frac{1-2x}{2x^2(1-x)^2} y = 0.$$

Solution by the PROPOSER.

Let $x = k^2$, $1-x = k'^2$, then, using JACOBI's notation for Elliptic Integrals, the solution of the equation is found to be

$$y = C_1 K^2 + C_2 K'^2 + C_3 K K',$$

where C_1, C_2, C_3 are arbitrary constants.

7257. (By Professor MALET, F.R.S.)—A cone of the second degree intersects a given sphere in two planes, one fixed, and the other touching in every position another fixed sphere concentric with the first; prove that the locus of the vertex of the cone is a pair of spheres.

Solution by the PROPOSER; Dr. CURTIS; and others.

Let the equation of the first fixed sphere be $x^2 + y^2 + z^2 - r^2 = 0$, of the fixed plane $x - a = 0$, and of the variable plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0,$$

where, by the conditions of the question, p is constant. The equation of the cone may be written in the form

$$\lambda(x^2 + y^2 + z^2) - 2(x-a)(x \cos \alpha + y \cos \beta + z \cos \gamma - p) = 0,$$

where λ is an indeterminate.

To determine the required locus, we have to eliminate λ , $\cos \alpha$, $\cos \beta$, $\cos \gamma$ from the equations

$$\lambda x - (x \cos \alpha + y \cos \beta + z \cos \gamma - p) - (x-a) \cos \alpha = 0,$$

$$\lambda y = (x-a) \cos \beta, \quad \lambda z = (x-a) \cos \gamma,$$

$$\lambda r^2 - a(x \cos \alpha + y \cos \beta + z \cos \gamma - p) = p(x-a).$$

Now, letting
$$x^2 + y^2 + z^2 - 2 \frac{r^2}{a} x + r^2 \equiv S,$$

it is found that the eliminant may be put in the form

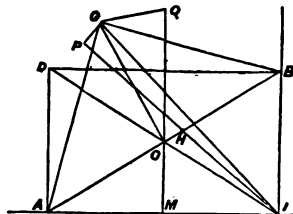
$$a^2(p^2 - a^2)S^2 + 4ap^2(r^2 - a^2)(x-a)S + 4p^2r^2(r^2 - a^2)(x-a)^2 = 0,$$

which is manifestly the equation of a pair of spheres.

6994. (By W. NICOLLS, B.A.)—A rigid uniform beam, resting on a rough horizontal plane and against an equally rough vertical wall, is prevented from slipping by a string attached to the beam and to the intersection of the planes; the beam lying in a vertical plane perpendicular to the wall, to find the tension and the limiting position of the string.

Solution by the PROPOSER; BELLE EASTON; and others.

Let $2a$ = length of the beam AB ,
 α = inclination of beam to the horizon,
 θ = inclination of string, ϵ = angle of friction, w = weight of beam, T = tension of string, O = centre of beam,
 H = point of attachment of string,
 Q = intersection of re-actions; then,
 if DA , DB be perpendiculars to the wall and plane, since $\epsilon = \angle CAD = \angle CBD$, the quadrilateral $ABCD$ is circumscribable by a circle, therefore $\angle ACB = \angle ADB = 90^\circ$; hence a circle passes through the points A , I , B , C , and D ; therefore $CO = AO = IO = BO = a$.



Again, $\angle CIA = \angle CBA = \alpha + \epsilon$, therefore $\angle CIP = \alpha + \epsilon - \theta$; and, since CI subtends the angle $90^\circ - \epsilon$ at the circumference, it is equal to

$$2a \sin(90^\circ - \epsilon) = 2a \cos \epsilon;$$

therefore $CP = CI \sin \angle CIP = 2a \cos \epsilon \sin(\alpha + \epsilon - \theta)$.

Again, $\angle COA = 2\alpha + 2\epsilon$, and $\angle AOQ = 180^\circ - \angle AOM = 90^\circ + \alpha$,
 therefore $\angle COQ = 90^\circ - \alpha - 2\epsilon$; hence $CQ = OC \sin \angle COQ = a \cos(\alpha + 2\epsilon)$.

Solution by the PROPOSER; KATE GALE; and others.

In *any* medium let a ray take a certain path from A to B; then, if its direction be reversed at B, it will follow the same path back to A.

And if from any point two rays start in opposite directions, each at right angles to the radius vector from the fixed point, they will describe similar and equal paths in a medium such as is mentioned above.

Therefore the parts of the path of any ray on each side of an apse are similar and equal, and the proposition follows.

[The emission theory explains refraction, and will guide us aright in a question like this. If, then, we imagine a particle to describe any path through this medium with (velocity always proportional to the refractive index and) *central force* depending only on the length of the radius vector, we have simply to mention well-known properties of apses in Particle Dynamics.]

7324. (By Professor WOLSTENHOLME, M.A.) — A triangle $A'B'C'$, whose angles are given, circumscribes a given triangle ABC ; prove that there will be two such triangles of maximum area; and, if these areas be Δ_1, Δ_2 , and R the radius of the circle ABC ,

$$\Delta_1 \sim \Delta_2 = 8R^2 \sin A \sin B \sin C,$$

$$\Delta_1 + \Delta_2 = 4R^2 (\sin^2 A \cot A' + \sin^2 B \cot B' + \sin^2 C \cot C').$$

[If the *sign* of the areas be reckoned according to the usual rule (that $A'B'C'$ is positive when, in going round the triangle in the order A', B', C' , the area is on the left hand), the smaller of these will be negative, and with this convention $\Delta_1 + \Delta_2 = 8R^2 \sin A \sin B \sin C$,

$$\Delta_1 - \Delta_2 = 4R^2 (\sin^2 A \cot A' + \dots + \dots).]$$

Solution by W. S. McCAY, M.A.

When $A'B'C'$ is a maximum, perpendiculars to its sides at A, B, C intersect in one or other of two fixed points O or O' (see Question 7067; also TOWNSEND'S *Modern Geometry*), at which the sides of $A'B'C'$ subtend angles $A \pm A', B \pm B', C \pm C'$, the upper sign being taken for the fixed point O that is inside the circumscribed circle $A'B'C'$, and the lower for the point O' that is outside. In the first case, the vertices $ABC, A'B'C'$ are in the same cyclical order; in the second, different. The points O, O' subtend equal or supplemental angles at the sides of ABC , and so are diametrically opposite on an equilateral hyperbola through ABC . The area of $A'B'C'$ is

$$\frac{1}{2} \sum \frac{ab \sin (C \pm C')}{\sin A' \sin B'}, \text{ which reduces to } \pm 2\Delta \cdot ABC + \frac{1}{2} \sum a^2 \cot A',$$

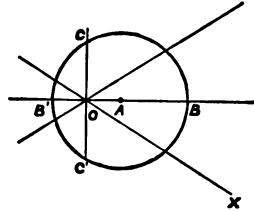
which gives the result in question.

If $A'B'C'$ and ABC are equiangular, O is at the orthocentre of ABC ; O' is any point on the circumscribed circle of ABC ; and $\Delta_1 = 4\Delta$ and $\Delta_2 = 0$.

in the circumference of the circle, the node becomes a cusp, and the curve a cardioid.

Solution by the PROPOSER.

Take the fixed point O as origin, the line joining it to the centre A of the circle as axis of x , and let the given line make an angle α with this axis. Let $OA = c$, radius of circle $= a$, and let ρ, ϕ be the polar coordinates of the point P on the locus corresponding to the point Q on the circle. Then in figure



$k \cdot OP = OQ^2$, and $\angle POX = 2 \angle QOX = 2\theta$.

The equation of the circle is $(x - c)^2 + y^2 = a^2$, and the coordinates of Q are $c + a \cos \beta, a \sin \beta$,

$$\text{therefore} \quad \frac{\sin(\theta - \alpha)}{a \sin \beta} = \frac{\cos(\theta - \alpha)}{c + a \cos \beta} = \frac{1}{OQ} = \frac{1}{(k\rho)^{\frac{1}{2}}};$$

$$\text{therefore} \quad a \sin \beta = (k\rho)^{\frac{1}{2}} \sin(\theta - \alpha) = (k\rho)^{\frac{1}{2}} \sin \frac{1}{2}(\phi - \alpha),$$

$$a \cos \beta = (k\rho)^{\frac{1}{2}} \cos(\theta - \alpha) - c = (k\rho)^{\frac{1}{2}} \cos \frac{1}{2}(\phi - \alpha) - c;$$

$$\text{therefore} \quad a^2 = k\rho - 2c(k\rho)^{\frac{1}{2}} \cos \frac{1}{2}(\phi - \alpha) + c^2,$$

$$\text{therefore} \quad (k\rho + c^2 - a^2)^2 = 2c^2 k\rho [1 + \cos(\phi - \alpha)];$$

or, in Cartesian coordinates, putting ρ^2 for $x^2 + y^2$,

$$[k^2 \rho^2 + (c^2 - a^2)^2 - 2c^2 k(x \cos \alpha + y \sin \alpha)]^2 = 4k^2 a^4 \rho^2.$$

Turning the axes through an angle α in the positive direction, this equation becomes

$$[k^2 \rho^2 - 2c^2 kx + (c^2 - a^2)^2]^2 = 4k^2 a^4 \rho^2 \dots\dots\dots (A);$$

therefore the locus of P is a Cartesian whose axis makes an angle α with OA, or OA bisects the angle between the axis and the given line OX.

If the point $(z, 0)$ be a focus, the line $y = i(x - z)$ must touch the curve.

Substituting for y in the equation of the curve, we get

$$4k^2 x^3 (kz - c^2)^2 + 4kx \{ (kz - c^2) [(c^2 - a^2)^2 - k^2 z^2] - za^4 kx \} + [(c^2 - a^2)^2 - k^2 z^2]^2 + 4a^4 k^2 z^2 = 0.$$

The condition for equal roots gives $z = 0$, or $k^2 z^2 - 2kz(c^2 - a^2) + (c^2 - a^2)^2 = 0$; therefore the origin is a focus, and the other two foci coincide at the point $\left(\frac{c^2 - a^2}{k}, 0 \right)$; therefore the curve is a limaçon.

Transferring the origin to the double focus, the equation becomes $[k(x^2 + y^2) - 2a^2 c^2(x^2 + y^2)]^2 = 4a^2 c^2(x^2 + y^2)$, or in polar coordinates

$$\rho = \frac{2a^2}{k} \cos \phi \dots \frac{2ac}{k}.$$

This is the ordinary form of the equation of Pascal's limaçon.

The node will be a crunode, cusp, or acnode, according as c is $<$ or $>$ or $= a$, that is, according as the fixed point O lies inside, upon, or outside the given circle. The existence of the node can also be proved directly from the

Solution by J. HAMMOND, M.A.

Taking the sides of the given angle α as axes, the equation to any line through a given point within it (a, b) is $y - b = m(x - a)$ (1), and, since the intercepts of this line on the axes are $a - \frac{b}{m}$ and $b - am$, the square of its length is $\left(\frac{am - b}{m}\right)^2 (1 + m^2 + 2m \cos \alpha) = \text{min.}$ (2). Differentiating (2) with respect to m , and removing the superfluous factor $2 \left(\frac{am - b}{m^3}\right)$, we have $am^3 + am^2 \cos \alpha + bm \cos \alpha + b = 0$ (3). Now, both a and b being positive, (3) has always one real negative root, and the value of this root substituted in (1) gives the equation to the minimum line, as is obvious from geometrical considerations.

When $\cos \alpha = 0$, we have to substitute $m = -\left(\frac{b}{a}\right)^{\frac{1}{2}}$ in (1), which then becomes $(x - a)a^{-\frac{1}{2}} + (y - b)b^{-\frac{1}{2}} = 0$, which touches $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}} + b^{\frac{1}{2}}$ at the point (a, b) .

7338. (By S. TEBAY, B.A.)—The parallel sides of a trapezoid are $m^2 + 2mn - n^2$ and $n^2 + 2mn - m^2$; show that it is divided into two equal trapezoids by the line $m^2 + n^2$.

Solution by W. W. TAYLOR, M.A.; R. KNOWLES, B.A.; and others.

The distances between the extreme lines and the mean are proportional to $2(mn - n^2)$ and $2m^2 - 2mn$; hence the areas are proportional to

$$2(m - n)n(2m^2 + 2mn) \text{ and } 2m(m - n)(2n^2 + 2mn),$$

and these are each equal to $4mn(m^2 - n^2)$; hence the two areas are equal.

7024. (By the EDITOR.)—If ACB be a circle on AB as diameter, and P, P' two points on the vector AC such that $CP = CP' = \text{distance of C from AB}$; show that (1) the locus of P is a heart-shaped curve circumscribing the circle, and of P' a continuous loop inscribed therein;

(2) the areas of these loci, in parts of the area of the circle, are $\frac{5}{4} \pm \frac{8}{3\pi}$;

7219. (By Professor SYLVESTER, F.R.S.) — Prove that a series of continually increasing integers, no one of which is a sum of positive multiples of all or any of those that precede it, cannot be continued *ad infinitum*.

Solution by the PROPOSER.

1. Suppose that one of the integers written down is a prime number p . Then, as is well known, if n be any other number of the series, $xp + yn$ may be made equal with positive values of x, y to any number not less than pn . Hence, the series in this case comes to an end as soon as pn is reached or surpassed.

2. Suppose the theorem to be true when one of the integers written down contains only i prime factors. Then I say it will be true if any one of the same contains $i + 1$ factors. For, suppose one of the numbers P to be

$$p^a \cdot q^b \cdot r^c \dots, \text{ where } a + b + c + \dots = i + 1.$$

It is obvious that only a finite number of terms of the series can contain either p , or q , or $r \dots$. For, if $p^a \cdot q^b \cdot r^c \dots$ and $pm, pm', pm'', \dots$ could be continued indefinitely, the same would be true of

$$p^{a-1} \cdot q^b \cdot r^c \dots, \quad m, m', m'', \dots,$$

contrary to the hypothesis. Let, then, n be any integer of the series not divisible by p , or q , or $r \dots$, i.e., prime to P ; then $xP + yn$ can be made equal to any number not less than Pn ; and consequently what remains of the series, after the previous different limited series are struck out, cannot be continued *ad infinitum*; and the theorem is true for the case where one of the numbers contains $i + 1$ factors, if it is true when one of the numbers contains only i factors; and, being true when i is unity, is true universally.

7166. (By Professor TOWNSEND, F.R.S.) — The central axis of a system of three forces in a common space being supposed to pass through the centre of the quadric determined by their three lines of action in the space; show, on elementary principles, that it must coincide with a principle axis of the surface.

Solution by the PROPOSER.

For since, on transference of the three forces to the centre of the quadric, the plane of their resultant moment is always conjugate to the direction of their resultant force with respect to the surface (see Quest. 6769); therefore, &c.

5250. (By the late Dr. BOOTH, F.R.S.) — Through a given point within an angle draw (1) the minimum line between the sides of the given angle; and (2) show that, when the angle is a right angle, the minimum line drawn through the given point is a tangent to the quadrantal hypocycloid passing through the same point.



MAR 23 1948

the tangent. The maximum value of t is found from $1 + 2x - 3x^2 - 4x^3 = 0$; and, dividing by $1 + x$ (evidently not a root), we obtain $4x^2 = x + 1$, $x = \frac{1}{2}(1 \pm \sqrt{17})$, giving as the evident solution

$$\sin \theta_1 = \frac{1}{2}(1 + \sqrt{17}) = .640393, \text{ \&c., whence } \theta_1 = P_2 AB = 39^\circ 49' 16''.$$

5. A cone cut from the sphere on AB as diameter, by the revolution of a line AP where $PAB = \theta$ round AB, has its entire surface given by

$$\text{curved surface} = 2a \cos \theta \cdot \pi a \sin 2\theta = 4a^2 \pi \sin \theta (1 - \sin^2 \theta),$$

$$\text{area of base} = (\pi a \sin 2\theta)^2 = 4a^2 \pi \sin^2 \theta (1 - \sin^2 \theta),$$

$$\text{therefore entire surface} = 4a^2 \pi (\sin \theta + \sin^2 \theta - \sin^3 \theta - \sin^4 \theta) = \pi t^2;$$

hence the surface is a maximum when t is a maximum, and the position of AP coincides with AP_2 .

7315. (By W. S. M'CAY, M.A.)—If XX' , YY' , ZZ' be the diagonals of a quadrilateral circumscribed to a circle, L, M, N their mid-points, and O the centre of the circle; prove that

$$\frac{OX \cdot OX'}{OL} = \frac{OY \cdot OY'}{OM} = \frac{OZ \cdot OZ'}{ON}$$

Solution by J. YOUNG, M.A.; Prof. NASH, M.A.; and others.

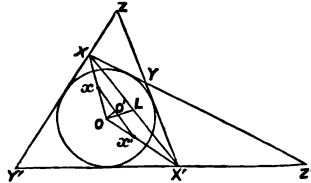
Take x, x' the inverse points to X, X' ; let O' be the middle point of xx' ; and join Oo' .

The triangles OXX' , $Ox'x$ being similar, we have $\frac{OX}{OL} = \frac{OX'}{OO'}$.

Hence it follows that

$$\frac{OX \cdot OX'}{OL} = \frac{Ox' \cdot OX'}{OO'} = \frac{R^2}{OO'}$$

But O' is also the middle point of yy' , and x, x' the inverses of YY', ZZ' ; for, since x, x', y, y', z, z' are the mid-points of the sides and diagonals of the inscribed quadrilateral, xx' and yy' are diagonals of a parallelogram, as are also xx' and zz' , and O' is their common mid-point, so that each of the given fractions is equal to $R^2 + Oo'^2$.



7185. (By MORGAN JENKINS, M.A.)—Show, as a special case of Gergonne's general construction for describing a circle to touch three given circles (SALMON'S *Conics*, Ch. ix., Art. 127), that the nine-points-circle of a triangle touches the three escribed circles of that triangle; also, in like manner, that it touches two of the escribed circles and the inscribed circle; show also that the three circles, each of which touches two of the escribed